East Texas Baptist University

School of Fine Arts

Department of Music

The Two-Part and Three-Part Inventions of Bach: A Mathematical Analysis

Honors Project

By Jennifer Shafer

Dr. Randall Sulton, Project Supervisor

Committee Members: Dr. Virginia Lile Boaz, Dr. Melissa Reeves

Marshall, Texas

March 2010

Introduction

Benoit B. Mandelbrot's book *The Fractal Geometry of Nature* was published in 1983. This publication updated and replaced two earlier works from 1975 and 1977 and codified his theory of fractal geometry as a new branch of mathematics. As a result of the creation of fractal geometry in the world of mathematics, music theorists have since created new techniques for analyzing music. These methods include study of fractal structure, fractal dimension, strange attractors in music, and Fourier analysis of musical lines.

For my own research in this area, I chose to analyze the fifteen Two-Part Inventions and fifteen Three-Part Inventions of Johann Sebastian Bach (1685-1750).[1] Each of these pieces can "be defined as a short contrapuntal work centering around the development of material from one or two motives".[2] While each piece has a unique motive, the compositional technique is the same, making all thirty pieces musically and technically different but similar in style. I based my analysis on two primary sources: "Fractal Geometry of Music," a 1993 article by Kenneth J. and Andreas J. Hsu, and *Fractals in Music: Introductory Mathematics for Musical Analysis*, published in 2007 by Charles Madden. Both sources suggested new methods of musical analysis, some of which I chose to use in my research.

This paper is divided into two main sections. The first section explains each concept as it is understood in the realm of mathematics, and then explains the applications for that concept to musical analysis. After discussion of the four concepts (fractal structure, fractal dimension, strange attractors, and Fourier analysis) and their applications to musical analysis, the second section of the paper begins by outlining the process that I developed to analyze the music using all four concepts. Following this is a summary of my findings in each of the four areas. Finally,

[1] The key difference between the Two-Part Inventions and the Three-Part Inventions is that the Two-Part Inventions have two independent lines, or voices, while the Three-Part Inventions are made up of three independent voices.
[2]Kent Kennan, *Counterpoint*, 4th ed. (Upper Saddle River, N.J.: Prentice-Hall, 1999), 126.

I discuss each piece individually, commenting on the computed dimensions, attractor plots, and spectral analysis graphs.

Fractal Structure

Mathematical Definition

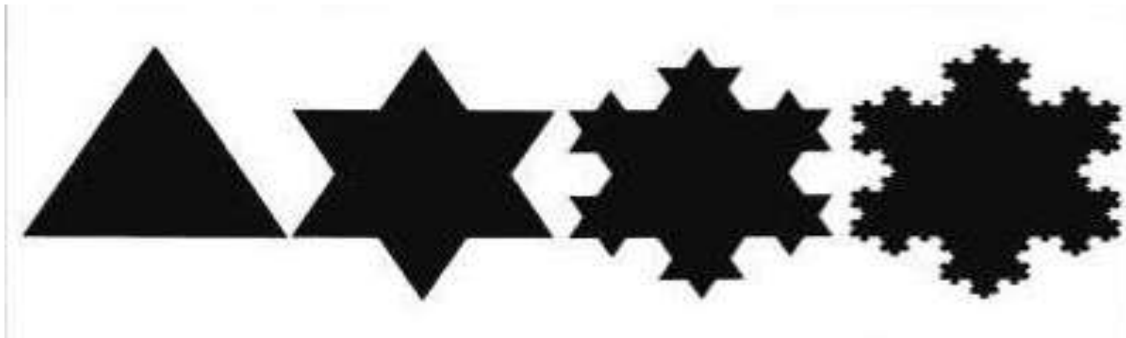
Benoit B. Mandlebrot, a pioneer in the field of fractal geometry, explains his reasons for developing fractal geometry in his book, *The Fractal Geometry of Nature.* Mandelbrot begins by discussing the inability of standard (Euclidean) geometry to describe the physical world by asking, "Why is geometry often described as 'cold' and 'dry?' One reason lies in its inability to describe the shape of a cloud, a mountain, a coastline, or a tree. Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth . . .".[3] Mandelbrot created his "geometry of nature" to correct this inadequacy. He explains that his fractal geometry "describes many of the irregular and fragmented patterns around, and leads to full-fledged theories, by identifying a family of shapes I call *fractals.*" Mandelbrot's fractals are defined both by their dimension and by their tendency to be *scaling,* or self-similar at all scales.

On the simplest level, fractals are generated by taking a basic "seed," or motive, and applying an iterative formula to the seed multiple times.[4] For example, the well-known fractal the Koch curve begins as an equilateral triangle. To produce this fractal, each side of the triangle is divided into three equal segments, the middle segment of each side is removed, and two segments equal to the length of the middle segment are put in its place to form another equilateral triangle. As this process is iterated over and over, the original equilateral triangle soon becomes the complex snowflake known as the Koch curve (See Fig. 1).[5]

---

[3] Benoit Mandelbrot, *The Fractal Geometry of Nature*, Rev. ed. (New York: W.H. Freeman and Company, 1983), 1.
[4] Larry Solomon, "The Fractal Nature of Music"; available from http://solomonsmusic.net/fracmus.htm; Internet; accessed 7 Feb 2008.
[5] Dietrick E. Thomsen, "Making Music—Fractally," *Science News*, Vol. 117, no. 12 (1980): 187.

Fig.1. The Koch Curve.[6]



Musical Application

In his article, “The Fractal Nature of Music,” Larry Solomon defines this idea of transforming a motive into a fractal as “the imitation, or translation, of a basic shape with attendant symmetry operations,” and further defines these symmetry operations as the basic mathematical operations of translation, reflection, and rotation. Solomon compares these mathematical operations to the musical operations of transposition (moving the musical motive up or down), retrograde (writing the motive backwards), augmentation (lengthening the motive by using longer note values), diminution (shortening the motive by using shorter note values), and inversion (turning the motive upside down), all of which are used frequently in contrapuntal composition. Contrapuntal textures result from “the combination of independent, equally important lines.”[7] In *The Analysis of Musical Form,* James Mathes defines imitation as “the repetition of material in a different voice or part at different times,” and states that imitation is often used “for elaborating musical ideas” and is frequently associated with contrapuntal genres.[8] The two variables in imitative writing are pitch interval (referring to distance between voices) and time interval (referring to the time distance between voice entrances).[9] According to

[6] Ibid.
[7] James Mathes, *The Analysis of Musical Form,* (Upper Saddle River, NJ: Prentice Hall, 2007), 274.
[8] Ibid.
[9] Ibid.

Solomon, this idea of transformation that is used to create fractals is also inherent in musical genres that are built on imitative contrapuntal techniques, such as canon and invention.

A canon is a "composition in which the voices enter successively at determined pitch and time intervals, all performing the same melody."[10] Thus, in a canon, all of the voices sing the same motive, but the individual voices' motives can be altered by any of the musical symmetry operations (See Fig. 2).

Fig. 2. Canon at the unison: Palestrina, *Illumina oculos meos*, mm. 1-9.[11]

As seen in Fig. 2, in a canon at the unison each voice enters on the same note. The entrance of each voice is marked by a red bracket, and the red arrows show that each voice is singing the same melody throughout the piece. Since they do not begin singing at the same time, the voices

---

[10] Donald J. Grout, and Claude V Palisca, *A History of Western Music*, 6th ed. (New York: W.W. Norton & Company, 2001), 791.
[11] Milo Wold and others, *An Outline History of Western Music*, 9th ed. (Boston: McGraw Hill, 1998), 46.

appear to be singing different melodies; however, the three black brackets and arrows show that the voices continue to sing the same lines throughout the piece with the same time delay as at the beginning. Thus, this simple canon was created from a single musical motive that was strictly imitated in three voices at the same pitch level, although at different time intervals.

Imitation is also used as a compositional technique in the genre of invention. An invention is similar to a canon, but, unlike the voices of a canon, the voices of an invention do not always use the same pitch interval throughout the piece. Instead, the invention is "based on a single theme, normally stated at the outset by each voice in succession, and reappearing at intervals throughout the piece."[12] This theme (called either a motive or a subject) is typically short, and is altered by means of contrapuntal devices throughout the piece (See Fig. 3).

---

[12] Douglass M. Green, *Form in Tonal Music: An Introduction to Analysis*, 2nd ed. (New York: Holt, Rinehart and Winston, 1979), 283.

Fig. 3. Analysis of Two-Part Invention: J.S. Bach, Invention No. 1 in C Major, mm. 1-10.[13]

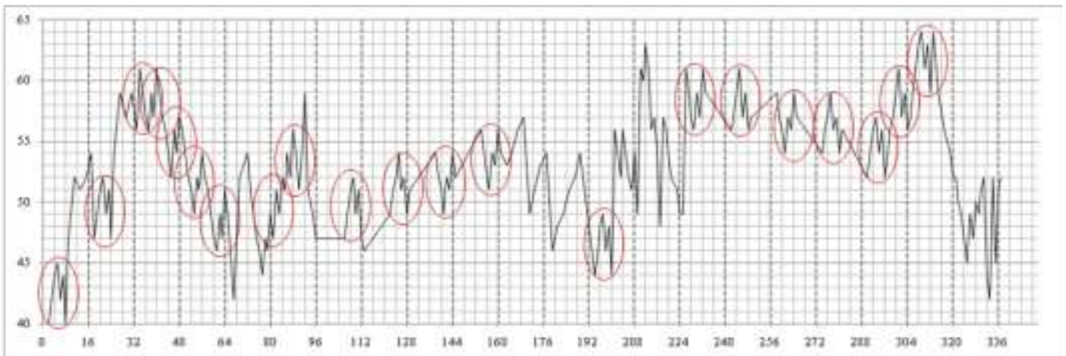
Fig. 3 shows a partial analysis of Johann Sebastian Bach's Two-Part Invention in C major by Walter Solomon. In this analysis, the opening subject, labeled **S,** is broken down into two motives, **a** and **b**. A third motive, **c**, which was derived from part of **b**, is also used, especially at cadential moments. In the same way that simple fractals can be constructed using a seed and basic symmetry operations, this invention is built almost entirely from the short subject introduced in the first measure.[14]

---

[13] Solomon.
[14] Solomon.

Madden’s method of creating a graph that shows the shape of the musical line by plotting pitch on the vertical axis versus time on the horizontal axis gives a visual representation of the musical line that makes the repeated entrances of the motive easier to see (See Fig. 4).

Fig. 4. Bach, Two-Part Invention No. 1, right hand line, subject entrances marked in red.



Fractal Dimension

Mathematical Definition

Similarity dimension

Our world is “made up of objects which exist in integer dimensions: single dimensional points, one dimensional lines and curves, two dimension plane figures like circles and squares, and three dimensional solid objects such as spheres and cubes.”[15] Sometimes, however, integers cannot accurately describe the dimensions of an object, since “dimension” is an approximate measurement of how much space a set fills. In *Fractal Geometry: Mathematical Foundations and Applications,* Kenneth Falconer explains that dimension is a “measure of the prominence of the irregularities of a set when viewed at very small scales.”[16] As explained by Solomon, “planet Earth is described in traditional science as a sphere or ellipsoid although its surface is not smooth, but rough . . . The coastline and boundary of India may be described in Euclidean

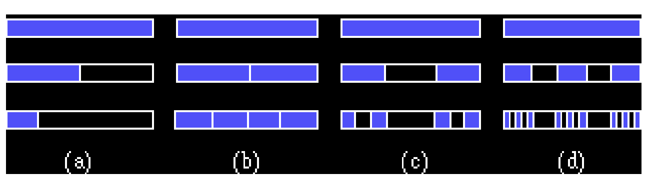
[15] “Fractals and Fractal Geometry,” Thinkquest; available from http://www.thinkquest.org/pls/html/think.library; internet; accessed 5 April 2008.
[16] Kenneth Falconer, *Fractal Geometry: Mathematical Foundations and Applications,* (Chichester, England: John Wiley & Sons, 1990), xxiv.

geometry as a triangle, but it is very complex and irregular."[17] Fractals function similarly. A fractal curve is not straight (like a line), so it has a dimension greater than one, but since a fractal curve does not completely fill the plane it occupies (like a square or circle) the curve has a dimension less than two. Therefore, a fractal curve has a dimension somewhere between one and two.[18] Consider again the simple fractal the Koch curve (See Fig. 1).

Fig. 1 shows only a few iterations of the Koch curve. The construction process can theoretically continue to infinity. As the iterations continue to infinity, however, the perimeter of the shape also approaches infinity as the length of each new side approaches zero. Despite the fact that the perimeter length of the shape becomes infinite, the total area the shape encloses remains finite. Since this concept makes no sense in the context of integer dimensions, "mathematicians had to redefine the term 'dimension' so that fractal dimensions could be calculated."[19]

Since many simple fractals are constructed by dividing a shape and retaining only some of the divided parts, one article suggests that dimension (fractional or integer) can be defined as D=[ln(number of pieces retained)]/[ln(number of pieces that the shape was divided into originally)][20] (See Fig. 5).

Fig. 5. Fractional Dimension.[21]



[17] Solomon.
[18] Fractals.
[19] Ibid.
[20] Fractals.
[21] Ibid.

Each line (a, b, c, and d) is divided into a specified number of pieces on each iteration, and then a specified number of the pieces are retained. For line (a), each line is divided into two segments, and one segment is retained, resulting in D=[ln(1)]/[ln(2)]=0. Since the retained segment is gradually shrinking, the final result as this process is repeated to infinity will be only a point, which will have a dimension of 0. For line (b), each piece is divided into two segments, and both are retained, giving a dimension of D=[ln(2)]/[ln(2)]=1. Since none of the pieces are discarded, the line will keep its standard dimension of 1. For line (c), each line is divided into thirds, and only two segments are kept. The dimension of this figure is D=[ln(2)]/[ln(3)]=.6309, meaning that line (c) has a fractional, or *fractal*, dimension. For line (d), the line is divided into fifths, and three segments are retained, resulting in a dimension of D=[ln(3)]/[ln(5)]=.6826. Line (d), then, also has a fractal dimension.[22]

This process can then be applied to "real" fractals, such as the Koch curve. Each segment of the Koch curve is divided into three segments, but since two new segments replace one of the original segments, there are actually four pieces retained. Using the same process, the dimension of the Koch curve is D=[ln(4)]/[ln(3)]=1.2619.[23] The same principle could also be applied to fractals which are "three-dimensional," which will have a dimension greater than two but less than three. This method of determining dimension is commonly called similarity dimension. Falconer states that "similarity dimension is meaningful only for a relatively small class of strictly self-similar sets."[24] Falconer further notes, however, that other methods of determining dimension, such as Hausdorff dimension and box-counting dimension, may be defined for any set.[25]

[22] Fractals.
[23] Ibid.
[24] Falconer, xxiv.
[25] Ibid.

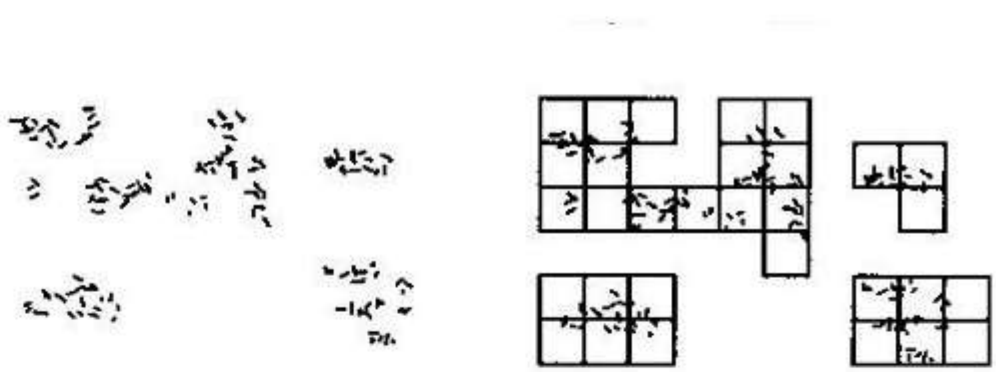
Box-counting dimension

Falconer defines the box-counting dimension for a set F as $\dim_B F = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$, where

$N_\delta(F)$=the smallest number of sets of at least diameter δ that can cover F (assuming that the set F being considered is both empty and non-bounded).[26] By mathematically manipulating this formula, Falconer determines that $N_\delta(F)$ is equivalent to the number of boxes of side length δ that intersect F. This result leads to the most common understanding of box-counting dimension: "To find the box

> dimension of a plane set F we draw a mesh of squares or boxes of side δ [see Fig. 6] and count the number $N_\delta(F)$ that overlap the set for various small δ (hence the name 'box-counting') . . . The number of mesh cubes of side δ that intersect a set is an indication of how spread out or irregular the set is when examined at scale δ. The dimension reflects how rapidly the irregularities develop as $\delta \to 0$".[27]

Fig. 6. A random set F with a box-counting grid.[28]



[26] Ibid., 42.
[27] Ibid., 43.
[28] Ibid., 40.

Musical applications

Overall dimension (Hsu)

According to research by Kenneth and Andreas Hsu, melodies of classical music may also have a fractal dimension. While Kenneth Hsu was studying natural catastrophes, he realized that there existed "an inverse log-log linear relation between the frequency (F) and a parameter expressing the intensity of the events (M) . . . and the relation can be stated by the simple equation: $F = c/M^D$."[29] Later, the Hsus realized that this was the relation that had "been called *fractal* by Mandelbrot, where c is a constant of proportionality and D is the fractal dimension."[30] The Hsus then applied this principle to classical music melodies.

Recognizing that musical notes alone do not create melodies, but rather the ordered succession of the notes, the Hsus determined that the succession of notes that make up a melody is fractal if "the incidence frequency (F) of the interval (i) between successive notes in a musical composition can be defined by the relation . . . $\log F = c - D \log i$, where c is a constant and D is the fractal dimension."[31]

Musicians define the spaces between musical notes as intervals, which can be measured by semitones, the smallest intervals in Western music (the distance from one note on the piano to its next closest neighbor). In their research, the Hsus determined the size of the interval between each successive note in the piece of music, and then determined the percentage frequency of each interval. The logs of the percentage values were graphed on a log-log linear plot, and the formula checked to see if the frequencies of the intervals in the melody had a fractal dimension. For this type of analysis, the Hsus combined the percentage frequencies from all of the voices to

[29] Kenneth J. Hsu and Andreas J. Hsu, "Fractal Geometry of Music," *Proceedings of the National Academy of Sciences of the United States of America,* Vol. 87, no. 3 (1990): 938.
[30] Ibid.
[31] Ibid., 938-9.

give an overall dimension, rather than considering the lines separately. The Hsus analyzed a total of six pieces of classical music and found all of the classical melodies to have a fractal dimension.

Melodic line dimension (Madden)

In his book *Fractals in Music: Introductory Mathematics for Musical Analysis* Charles Madden proposes a different method of measuring the dimension of a piece of music. Madden's method is based on the box-counting method and uses graphs showing the shape of the music, created by plotting pitch versus time.[32] By applying a grid to the graph, the boxes can be counted to determine the dimension. Since this method becomes tedious for a piece of any length, Madden determined a formula that sums the difference between the pitches, thus giving a count of the boxes that are crossed by the lines in the graph. Madden points out that the problems of accuracy in the box-counting method at small values of δ are non-existent in this type of analysis, since the musical examples cannot go beyond the individual note space. He thus simplifies the process. Madden's method must be applied to separate lines individually, since this is a measure of the dimension of the melodic line.

Attractors

Mathematical Definition

Dynamical systems are systems whose status will change over time, and the term chaos "is usually reserved for dynamical systems whose state can be described with differential equations in continuous time or difference equations in discrete time."[33] In "Chaos, Fractals, and Statistics," Sangit Chatterjee states that an accurate and complete description of a dynamical

---

[32] Charles Madden, *Fractals in Music: Introductory Mathematics for Musical Analysis,* 2nd ed. (Salt Lake City: High Art Press, 2007), 143.
[33] Sangit Chatterjee and Mustafa R. Yilmaz, "Chaos, Fractals and Statistics", *Statistical Science,* Vol. 7, no. 1 (1992): 52.

system requires the ability to describe the system at a specific point in time as well as describe the behavior of the system over a given time interval.[34] This task is accomplished by examining the system in terms of its *phase space*. In the *phase space* of a system, “points in this space represent instantaneous descriptions of system status at different points in time.”[35] The closely related concept of *phase plane* can be used in differential equations to plot solutions to ordinary differential equations of the form $\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}$, which is referred to as the phase plane equation. For this equation, which is dependent on the *x* and *y* variables, the *xy*-plane is called the *phase plane*.[36] Thus, the phase plane is a two dimensional plane in which the solutions to the system of equations can be plotted. When this concept is extended beyond two dimensions, the term *phase space* is used, regardless of the number of dimensions used in the system. Chatterjee illustrates this by considering a single particle that is in motion in three-dimensional space. At an instantaneous moment in time, Chatterjee states that six coordinates will be required to describe the position of the particle: three coordinates that show the current position of the particle, and “three additional momentum coordinates showing the rate of change in the position coordinates”.[37] Thus, while the particle is in motion in standard three-dimensional space, the phase space of the single particle is six-dimensional.[38]

In his article “Chaos and Ecology: Is Mother Nature a Strange Attractor?” Alan Hastings defines an attractor as “a set of points in phase space . . . that represent a stable set of final

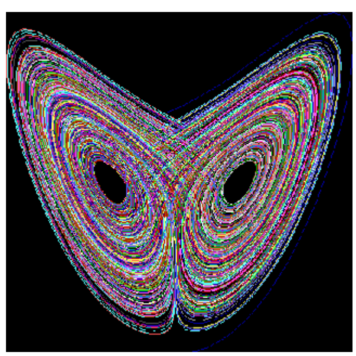
---

[34] Ibid., 53.
[35] Ibid., 53-4.
[36] Kent B. Nagle, Edward B. Saff, and Arthur David Snider, *Fundamentals of Differential Equations*, 6th ed. (Boston: Pearson, 2003), 265.
[37] Chatterjee, 54.
[38] Ibid.

dynamics for the system".[39] He further explains that the attractor is final in three ways: "First, once the state of the system or model is in this set, it does not leave this set. Second, all points of the set are reached. Finally, any trajectory starting near enough to the attractor approaches the attractor."[40] Hastings points out that within a chaotic system there typically exists an attractor to which any solution will converge, provided that the solutions are sufficiently close to the attractor initially. He also states that chaotic systems often have strange attractors that can often be identified by their "twisted" shapes and can be more accurately identified by their fractional dimension.

Madden defines three types of attractors: fixed, periodic, and strange. According to Madden, a fixed attractor is a single point towards which motion tends, such as the center of a spiral. A periodic attractor occurs when motion does not tend towards a single point, but rather oscillates through a set of points, like a sine wave. A strange attractor is defined as a "periodic attractor that is smeared out so that the orbits never repeat exactly".[41] The Lorenz attractor is an example of a strange attractor (See Fig. 7).

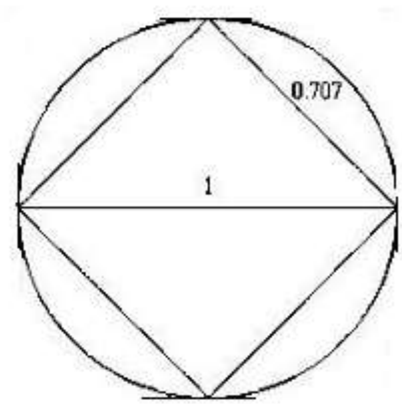
Fig. 7. Lorenz attractor.[42]



[39]Alan Hastings, Carole L. Hom, Stephen Ellner, Peter Turchin, H. and Charles J. Godfray., "Chaos in Ecology: Is Mother Nature a Strange Attractor?" *Annual Review of Ecology and Systematics,* Vol. 24 (1993): 5.
[40] Ibid.
[41] Madden, 6.
[42] Fractals.

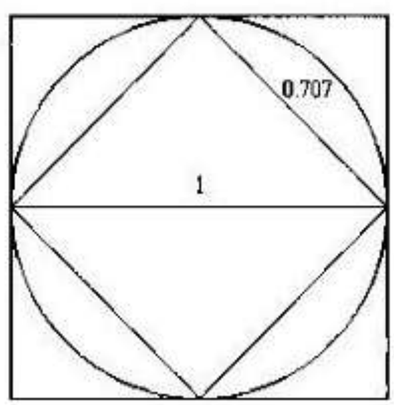
Madden further explains the concept of an attractor with an attempt to find the exact value of π using the exhaustion method proposed by Archimedes.[43] He begins by inscribing a square inside a circle with diameter 1 (See Fig. 8), and then calculating the perimeter of the inscribed square as approximately 2.828.

Fig. 8. Circle with inscribed square.[44]



Madden continues by circumscribing the circle with another square (See Fig. 9), the perimeter of which is equal to 4.

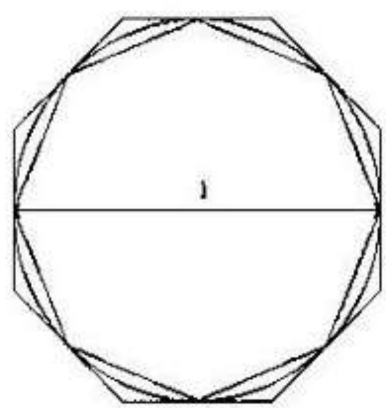
Fig. 9. Circle with inscribed and circumscribed squares.[45]



[43] Ron Larson, Robert Hostetler, and Bruce H. Edwards, *Calculus: Early Transcendental Functions*, 4th ed. (Boston: Houghton Mifflin, 2007), 297.
[44] Madden, 40.
[45] Ibid.

Madden points out that the perimeter of the circle must be somewhere between the perimeters of the squares and estimates it at (2.828+4) / 2 ≈3.414. While this number is not very close to the actual decimal approximation of π (3.14159 . . .), this method allows a closer approximation to the actual value of π by continuing to circumscribe and inscribe the circle with polygons whose sides are closer to the circle by using polygons with a greater number of sides (See Fig. 10).

Fig. 10. Circle with inscribed and circumscribed octagon.[46]



Madden explains that as this process is continued, the perimeters will approach the circle more closely, and a limit will be reached when the polygons have "an infinite number of infinitesimally small sides, lying an immeasurably small distance from the circle, giving us a perimeter infinitely close to the true value of π."[47]

As this process is repeated, the values found for π on each iteration "swing positively and negatively from side to side in a smaller and tighter trajectory, and could be said to spiral in on the limit. The center of this convergence (the circle itself) is the attractor."[48] Thus, this *attractor* was the point towards which the motion moved as the approximations moved closer and closer to π.

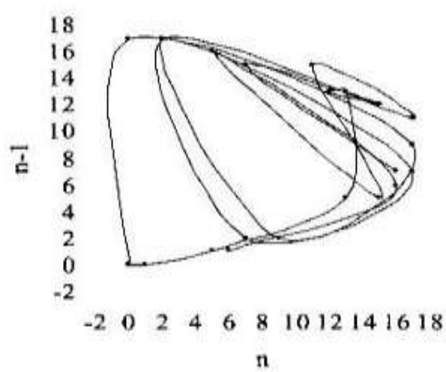
---

[46] Ibid.
[47] Ibid., 39-40.
[48] Ibid., 40.

Musical application

Madden applies this concept to music by creating graphs that are intended to show the presence of any attractors in the music. These graphs, which consist of scatter plots of the notes connected by smooth lines, are created by plotting the numerical "values" of the notes on the vertical axis against the same values plotted on the horizontal axis, but delayed by one note (value (*n*-1) against value *n*).[49] This effectively traces the path that the notes follow throughout the piece. Madden includes several examples of these graphs showing strange attractors in music (See Figs. 11 and 12).

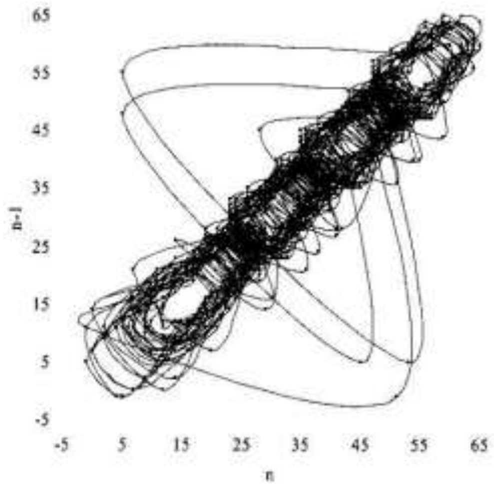
Fig. 11. Attractor graph: Ian Stewart, chaotic music.[50]



Madden explains that "this random structure is a single orbit that will never repeat and should be understood as a strange attractor".[51]

[49] Ibid., 50
[50] Ibid., 53.
[51] Ibid., 52.

Fig. 12. Attractor graph: Frederic Chopin, *Etude Op. 10 No. 1*.[52]



Although this plot lies strongly along the diagonal line (showing a strong sense of key), Madden states that since the music never repeats, this piece is also an example of a strange attractor.[53]

Fourier Analysis

Mathematical Definition

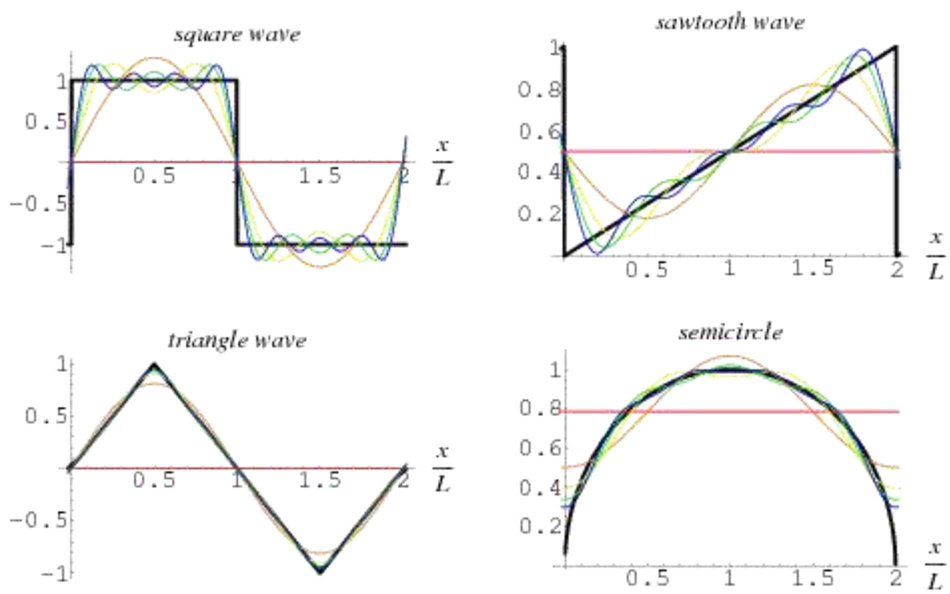
Fourier analysis, named for Jean Baptiste Joseph, Baron de Fourier (1768-1830), is used to break down a complex curve into a series of simpler curves. While studying the motions of heat waves flowing through objects, Fourier discovered that any periodic wave, regardless of its complexity, can be written as a sum of many simple waves.[54] The sum of simple waves (made up of sines and cosines) represents an expansion of the original periodic wave and is called a Fourier series.[55] Once the Fourier series is obtained, the simple waves can be “plugged in, solved individually, and then recombined to obtain the solution to the original problem or an approximation to it to whatever accuracy is desired or practical.”[56] Thus, Fourier analysis allows

---

[52] Ibid., 56.
[53] Ibid.
[54] Transnational College of Lex, *Who is Fourier? A Mathematical Adventure.* trans Alan Gleason. (Boston: Language Research Foundations, 1995), 11.
[55] Eric W. Weisstein, "Fourier Series," *MathWorld*--A Wolfram Web Resource; internet; accessed 1 October 2009.
[56] Ibid.

complex waves, for which there may be either a complex formula, or no known formula, to be approximated as a sum of simple waves, which can then be studied individually (See Fig. 13).

Fig. 13. Complex waves and their Fourier approximations.[57]



The general form for a Fourier series is: $f(\theta) = \frac{1}{2}2a_0 + \sum_{n=1}^{\infty}(a_n \cos(n\theta) + b_n \sin(n\theta))$. In this equation, the a_n and b_n terms are constants. In *Music: A Mathematical Offering*, David J. Benson explains how the a_n and b_n terms can be found using the following formulas, which are derived from the addition formulas for sine and cosine:[58] $\int_0^{2\pi} \cos(m\theta)\sin(n\theta)\,d\theta = 0$;

$$\int_0^{2\pi} \cos(m\theta)\cos(n\theta)\,d\theta = \begin{cases} 2\pi, & \text{if } m = n = 0, \\ \pi, & \text{if } m = n > 0, \\ 0 & \text{otherwise} \end{cases}; \text{ and } \int_0^{2\pi} \sin(m\theta)\sin(n\theta)\,d\theta = \begin{cases} \pi & \text{if } m = n = 0, \\ 0 & \text{otherwise} \end{cases}.$$

To find the coefficient a_m, Benson multiplies $f(\theta)$ by cos$(m\theta)$ and then integrates the result by considering separately each part of the sum of the integral (assuming m>0).

[57] Ibid.
[58] David J. Benson, *Music: A Mathematical Offering*, (New York: Cambridge University Press, 2007), 39.

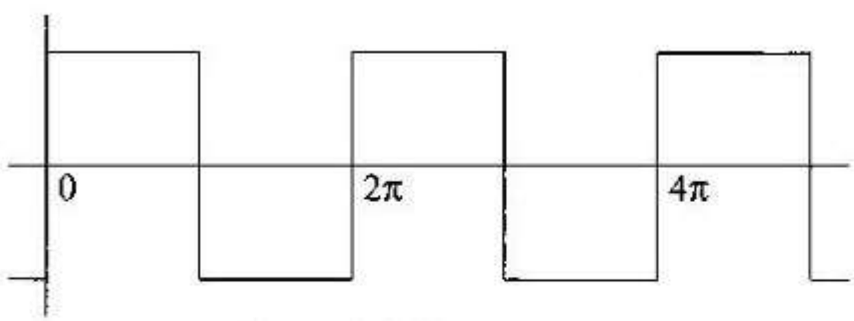
$$\int_0^{2\pi}\cos(m\theta)f(\theta)\,d\theta=\int_0^{2\pi}\cos(m\theta)\left(\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos(n\theta)+b_n\sin(n\theta)\right)\right)d\theta=$$

$$\frac{1}{2}a_0\int_0^{2\pi}\cos(m\theta)\,d\theta+\sum_{n=1}^{\infty}\left(a_n\int_0^{2\pi}\cos(m\theta)\cos(n\theta)\,d\theta+bn\int_0^{2\pi}\cos(m\theta)\sin(n\theta)\,d\theta\right)=\pi a_m.$$

The first term of the expanded integral, $\frac{1}{2}a_0\int_0^{2\pi}\cos(m\theta)\,d\theta$, will be equal to 0 due to the periodic nature of the cosine function from 0 to 2π. Since *m*>0, the $a_n\int_0^{2\pi}\cos(m\theta)\cos(n\theta)\,d\theta$ portion of the expanded integral will have only one nonzero term (when *m* = *n* >0, the value of the integral will be equal to π, and $a_n = a_m$). The third part of the integral, $bn\int_0^{2\pi}\cos(m\theta)\sin(n\theta)\,d\theta$, will also be equal to 0, thus giving the result $\int_0^{2\pi}\cos(m\theta)f(\theta)\,d\theta=\pi a_m$. It follows that, for *m* >0, $a_m=\frac{1}{\pi}\int_0^{2\pi}\cos(m\theta)f(\theta)\,d\theta$. By a similar process, for m >0, $b_m=\frac{1}{\pi}\int_0^{2\pi}\sin(m\theta)f(\theta)\,d\theta$.[59]

Benson illustrates with an example of determining Fourier coefficients for the square wave, defined by $f(\theta) = 1$ for $0 \le \theta < \pi$ and $f(\theta) = -1$ for $\pi \le \theta < 2\pi$ (see Fig. 14).

Fig. 14. The square wave[60]



[59] Ibid., 40.
[60] Ibid., 41.

To find the Fourier coefficients for this equation, Benson uses his formulas for a_m and b_m to obtain the following result (there are two integrals in each case because the function $f(\theta)$ is a piecewise function).[61]

$$a_m = \frac{1}{\pi}\left(\int_0^{\pi}\cos(m\theta)\,d\theta - \int_{\pi}^{2\pi}\cos(m\theta)\,d\theta\right)$$

$$= \frac{1}{\pi}\left(\left[\frac{\sin(m\theta)}{m}\right]_0^{\pi} - \left[\frac{\sin(m\theta)}{m}\right]_{\pi}^{2\pi}\right) = 0$$

$$b_m = \frac{1}{\pi}\left(\int_0^{\pi}\sin(m\theta)\,d\theta - \int_{\pi}^{2\pi}\sin(m\theta)\,d\theta\right)$$

$$= \frac{1}{\pi}\left(\left[-\frac{\cos(m\theta)}{m}\right]_0^{\pi} - \left[-\frac{\cos(m\theta)}{m}\right]_{\pi}^{2\pi}\right)$$

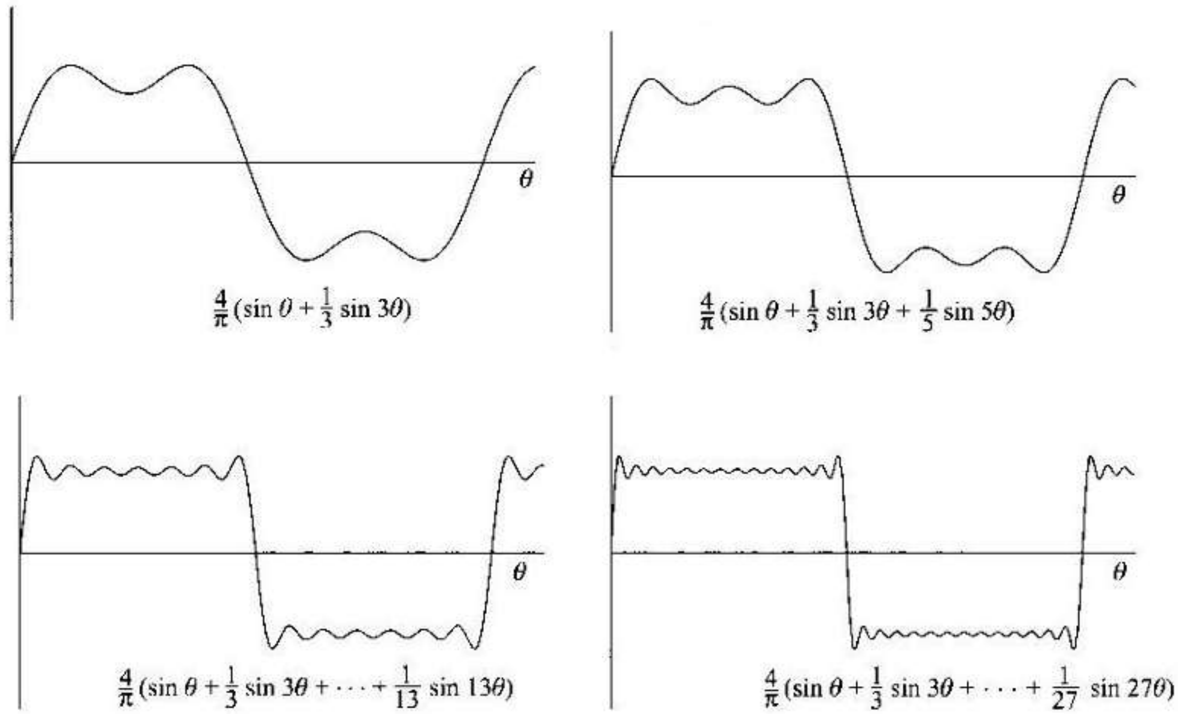
$$= \frac{1}{\pi}\left(-\frac{(-1)^m}{m} + \frac{1}{m} + \frac{1}{m} - \frac{(-1)^m}{m}\right) = \begin{cases} \dfrac{4}{m\pi} & (m \text{ odd}) \\ 0 & (m \text{ even}) \end{cases}$$

In a simplified format, the Fourier series for a square wave is $\frac{4}{\pi}\left(\sin\theta + \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta + \ldots\right)$.

Benson gives a diagram of the first few terms in this series to show the “square” curve that is obtained by this approximation (See Fig. 15).

[61] Ibid., 42.

Fig. 15. Square curve approximations.[62]



The overshoot at each turning of the curve is due to the fact that the function is everywhere defined but is not continuous. This "ringing" that occurs at discontinuities is known as the Gibbs phenomenon.[63] Despite this overshoot at points of discontinuity, the approximation comes close to duplicating the original curve, and the formulas for the constituent curves are more manageable than the original formula.

Musical Application

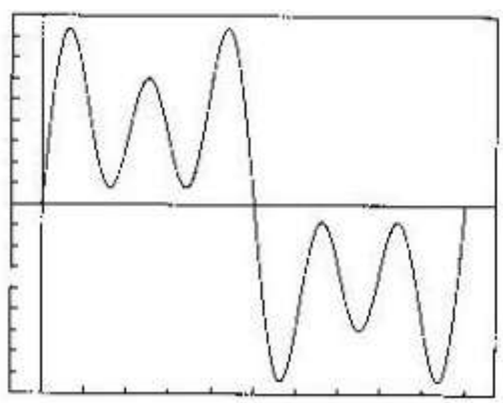
The final chapter in Madden's book gives direction for using Fourier analysis with graphs of the musical shape to break down the complex curve dictated by the musical line into the constituent simple curves. Madden's method does not include an analysis of the time factor, but is strictly one-dimensional, examining the melodic shape of the music with no reference to

[62] Ibid., 43.
[63] Weisstein.

time.[64] By creating an inverse transform, the coefficients in the Fourier series can be checked for accuracy, since the inverse transform should look similar to the original curve.

Madden gives the equation for a waveform as $f(t) = A_0 + A_1\cos 1\omega t + A_2\cos 2\omega t + \ldots$ $B_1\sin 1\omega t + B_2\sin 2\omega t \ldots . = A_0 + \Sigma(A_n\cos n\omega t + B_n\sin n\omega t)$.[65] The A_n and B_n terms can be determined using the formulas $A_n = \frac{2}{T}\sum_{t=0}^{T-1} f(t)\cos\frac{2\pi nt}{T}$ and $B_n = \frac{2}{T}\sum_{t=0}^{T-1} f(t)\sin\frac{2\pi nt}{T}$. In these formulas, T=number of samples (number of notes in the piece), $f(t)$=the value of the curve at a specific time t, and n=the harmonic number. Madden explains this process for the complex wave given by the equation $f(t) = 5\sin 2\pi 100t + 3\sin 2\pi 300t + 4\sin 2\pi 500t$ (See Fig. 16).

Fig. 16. Complex wave.[66]

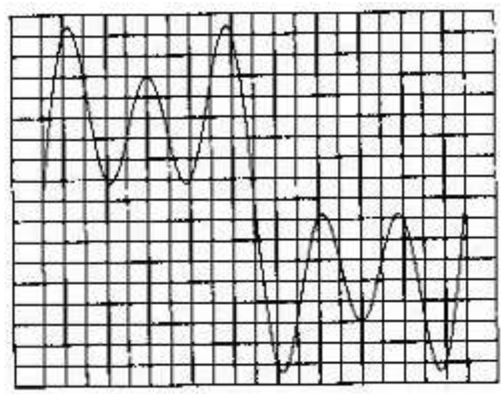


In order to treat this graph like a graph of a melodic line, Madden superimposes a grid over the curve in order to estimate values for $f(t)$ at specific values of t (see Fig. 17).

---

[64] Charles Madden, e-mail message to author, September 2, 2009.
[65] Madden, *Fractals,* 196.
[66] Ibid., 200.

Fig. 17. Waveform with superimposed grid.[67]



Madden then estimates values for *f(t)* and applies his formulas to find the A_n and B_n terms. After finding the A_n and B_n numbers, Madden uses these numbers to find the C_n terms (See Figs. 18 and 19).[68]

[67] Ibid., 201.
[68] Note: These tables are reduced in size. The actual tables would have more columns for A_n and B_n. They would run through n=10, since the number of samples was 20. See Appendix A.

Fig. 18. Values for A_n terms.[69]

t	$f()$	A_1	A_2	A_3	A_4	A_5
$f(t)\cos(2\pi nt/20)$		$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
1.00000	7.80000	7.41824	6.31033	4.58471	2.41031	-0.00003
2.00000	5.50000	4.44959	1.69958	-1.69962	-4.44961	-5.50000
3.00000	0.90000	0.52901	-0.27812	-0.85595	-0.72811	0.00001
4.00000	3.00000	0.92704	-2.42706	-2.42704	0.92708	3.00000
5.00000	5.80000	-0.00002	-5.80000	0.00006	5.80000	-0.00011
6.00000	2.90000	-0.89616	-2.34613	2.34617	0.89610	-2.90000
7.00000	0.80000	-0.47023	-0.24721	0.76084	-0.64722	0.00002
8.00000	5.60000	-4.53051	1.73056	1.73040	-4.53042	5.60000
9.00000	7.90000	-7.51336	6.39130	-4.64363	2.44143	-0.00026
10.00000	-0.20000	0.20000	-0.20000	0.20000	-0.20000	0.20000
11.00000	-8.10000	7.70354	-6.55296	4.76090	-2.50279	-0.00033
12.00000	-5.90000	4.77317	-1.82310	-1.82335	4.77332	-5.90000
13.00000	-1.10000	0.64656	0.33994	-1.04617	0.88989	0.00005
14.00000	-3.10000	0.95792	2.50799	-2.50790	-0.95807	3.10000
15.00000	-6.20000	-0.00007	6.20000	0.00020	-6.20000	-0.00034
16.00000	-3.10000	-0.95799	2.50791	2.50802	-0.95781	-3.10000
17.00000	-1.10000	-0.64657	0.33989	1.04615	0.88995	0.00007
18.00000	-6.00000	-4.85415	-1.85425	1.85388	4.85392	6.00000
19.00000	-8.10000	-7.70359	-6.55317	-4.76133	-2.50347	-0.00057
	Total	0.03240	-0.05452	0.02635	0.20450	0.49852
×2/20	A_n	0.00324	-0.00545	0.00264	0.02045	0.04985

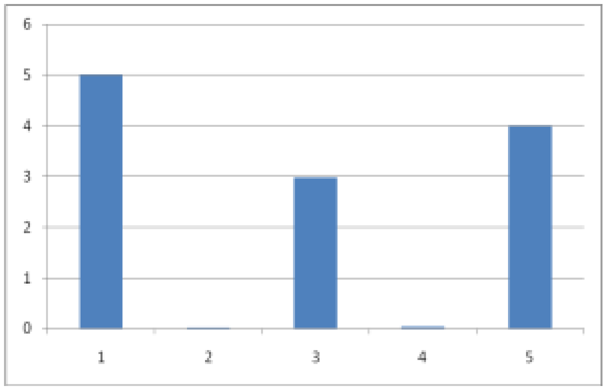
Fig. 19. Values for B_n and C_n terms.[70]

t	$f(t)$	B_1	B_2	B_3	B_4	B_5
$f(t)\sin(2\pi nt/20)$		$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
1.00000	7.80000	2.41034	4.58473	6.31034	7.41825	7.80000
2.00000	5.50000	3.23283	5.23082	5.23080	3.23279	-0.00004
3.00000	0.90000	0.72812	0.85595	0.27811	-0.52901	-0.90000
4.00000	3.00000	2.85317	1.76334	-1.76338	-2.85316	0.00004
5.00000	5.80000	5.80000	-0.00004	-5.80000	0.00009	5.80000
6.00000	2.90000	2.75806	-1.70460	-1.70455	2.75808	-0.00006
7.00000	0.80000	0.64721	-0.76085	0.24723	0.47021	-0.80000
8.00000	5.60000	3.29157	-5.32590	5.32595	-3.29170	0.00016
9.00000	7.90000	2.44118	-4.64342	6.39114	-7.51328	7.90000
10.00000	-0.20000	0.00000	-0.00000	0.00000	-0.00001	0.00001
11.00000	-8.10000	2.50310	-4.76117	6.55315	-7.70364	8.10000
12.00000	-5.90000	3.46798	-5.61127	5.61119	-3.46776	-0.00026
13.00000	-1.10000	0.88992	-1.04616	0.33989	0.64660	-1.10000
14.00000	-3.10000	2.94829	-1.82208	-1.82221	2.94824	0.00016
15.00000	-6.20000	6.20000	0.00014	-6.20000	-0.00027	6.20000
16.00000	-3.10000	2.94826	1.82219	-1.82205	-2.94832	-0.00018
17.00000	-1.10000	0.88991	1.04617	0.33996	-0.64652	-1.10000
18.00000	-6.00000	3.52665	5.70629	5.70641	3.52697	0.00040
19.00000	-8.10000	2.50293	4.76088	6.55284	7.70342	8.10000
20.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
	Total	50.03952	0.09503	29.77483	-0.24904	40.00023
×2/20	B_n	5.00395	0.00950	2.97748	-0.02490	4.00002
$\sqrt{A_n^2 + B_n^2}$	C_n	5.00395	0.01096	2.97749	0.03222	4.00033

The C_n terms are then used to create the spectral analysis (see Fig. 20).

[69] Ibid., 202.
[70] Ibid., 203.

Fig. 20. Spectral analysis.

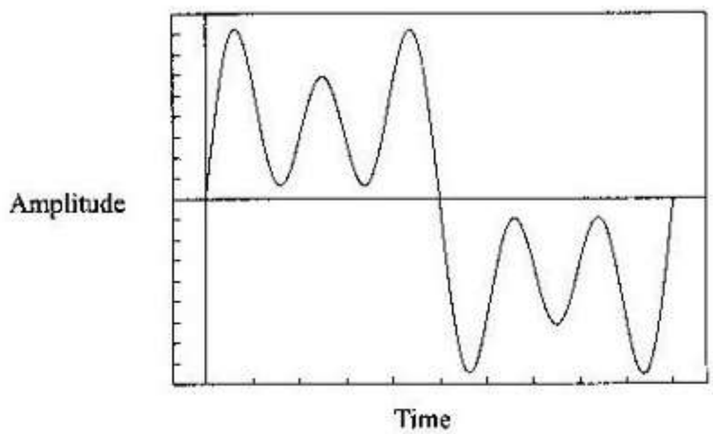


As expected from the original formula, the amplitudes at 1, 3, and 5 are 5, 3, and 4, respectively. The small amplitudes at 2 and 4 are due to errors in the estimates.[71] This type of analysis, when applied to a melodic line, shows "the amplitues of the sine and cosine waves that make up the complicated waveform that is the melodic shape."[72]

The final step in this type of analysis is to create an inverse transform using the equation for a waveform and the A_n and B_n terms found by using the spreadsheets to reconstruct the original graph, thus checking the accuracy of the coefficients. This reconstructed graph has a slight negative bias due to inaccuracies in some of the estimates, but the reconstructed graph is nearly identical to the original graph (see Fig. 21).

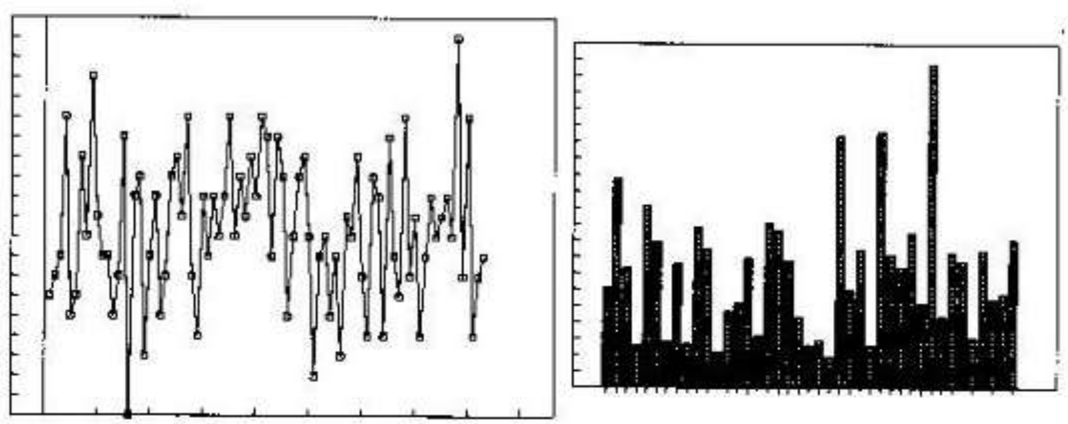
[71] Ibid., 203.
[72] Ibid., 204.

Fig. 21. Reconstructed Waveform.[73]



When this analysis process is applied to a melodic line, the process is different only in that the samples are taken at each note, thus eliminating the problem of inaccurate estimates. This type of analysis is useful in determining the relative simplicity or complexity (angularity or "randomicity") of the melodic lines. The amplitudes of the lower harmonics are typically high, while in most pieces the middle and upper harmonics generally have smaller amplitudes. Madden includes several examples of pieces that have higher amplitudes in the middle and upper harmonics (see Figs. 22, 23, 24).

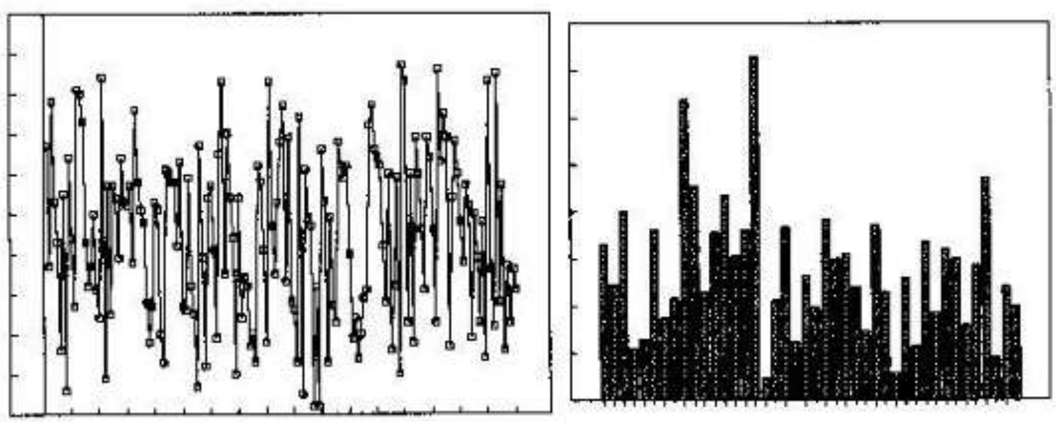
Fig. 22. Waveform and Spectrum: Richard Voss, white music.[74]



[73] Ibid., 205.
[74] Ibid., 211.

Madden comments that this "waveform is very angular; the spectrum contains much energy in the higher harmonics . . . [T]he spike at the 33rd harmonic is among the higher frequencies that cause the extreme angularity of the waveform."[75]

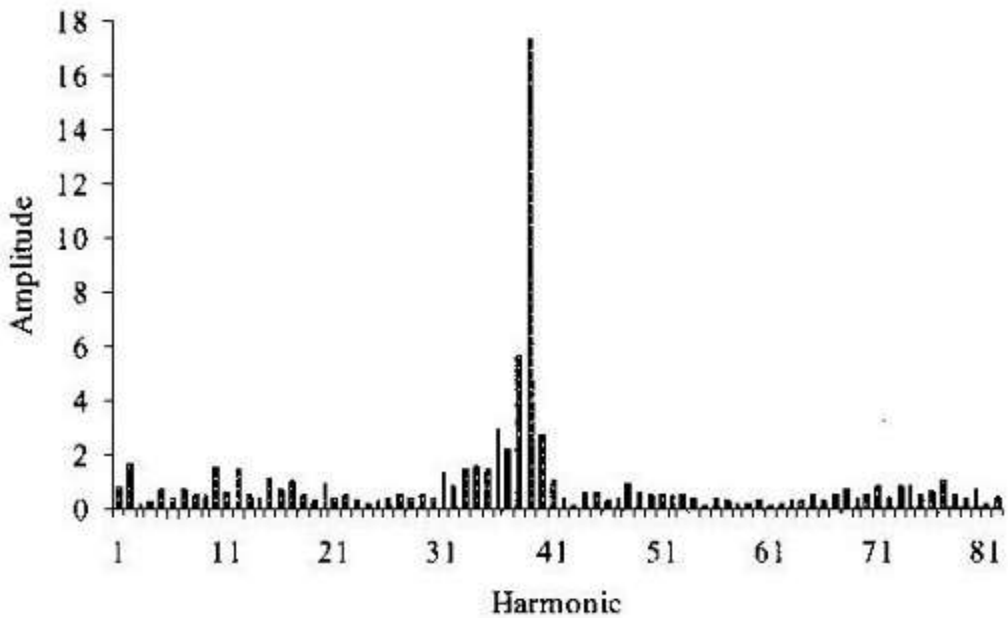
Fig. 23: Waveform and Spectrum: Iannis Xenakis, *Eonta.*[76]



Madden states that this waveform is also "very angular, and the frequencies look a little more like a uniform distribution, although there are some notable gaps and peaks."[77]

[75] Ibid., 210.
[76] Ibid., 212
[77] Ibid., 211.

Fig. 24: Spectrum: Chopin, *Etude Op. 10 No. 1.*[78]



Madden explains the reason for the single large harmonic that dominates the graph. This etude is "almost entirely a single sinusoid at the 39th harmonic; that is, it is a two-measure unit repeated almost verbatim 39 times."[79]

Development of Analysis Process

When I initially started work on this project, I began working with the Hsus' method of measuring dimension. To implement their process, I determined each individual interval in the invention and used tally marks to count them. Quickly recognizing the possibility of error allowed by this method, and after noting the use of pitch numbering in Madden, I set up an Excel spreadsheet and used Excel's capabilities to calculate the intervals, count their occurrences, calculate the percentage frequencies, create the graph, and determine the dimension of the graph. For each piece I only needed to input the pitch numbers for the notes in the music.

---

[78] Ibid., 219.
[79] Ibid., 219.

I then added Madden's method of creating a graph that plots pitch versus time. Since I had already set up the spreadsheets to input pitch, I then added two more columns to measure the time value of each note. After some trial and error, I determined that the simplest way to measure time would be to record all the notes in terms of the smallest note value that is used consistently throughout the piece, typically a sixteenth note. In one column I entered the number of sixteenth notes that the note was worth (1 for a sixteenth note, 2 for an eighth note, 4 for a quarter note, etc.), and in the second column I set up a formula to sum the time values cumulatively. By setting up both columns, I instituted a check system, since at the end of each measure, the cumulative number had to be a multiple of the time signature by the base note value. I now had to input the pitch numbers and the time value of each note.

Next I added Madden's modified version of the box-counting method to measure the dimension of the music. For this step, I only needed to add the formulas since I had already input all the pitches and all of the intervals were already being measured. I then added Madden's graphs to show the presence of attractors. Since I already had the individual notes in each spreadsheet, I simply added this graph to each file.

Finally, after following the examples in the book and corresponding by e-mail with the author, I was able to add the necessary formulas to my spreadsheets to perform Madden's technique of Fourier analysis and create a graph showing the spectral distribution of the harmonics. I analyzed each voice individually, since this type of analysis studies the "shape" of a melodic line rather than how the two voices work together. While the spectral analysis is the part of the analysis that is meaningful, I created the inverse graphs for each voice in order to check my work on the previous spreadsheets—the inverse graphs would not look like the originals if I made any mistakes in my formulas or entries.

Summary of Findings

A brief summary of my findings in each of the four areas (structure, dimension, attractors, and harmonic analysis) follows, along with a summary for each individual piece. For each piece I briefly discuss the dimensions—measured using both methods—and then include the following graphs: dimension (Hsu), attractor (Madden), spectral analysis (Madden), and reconstructed waveforms (Madden). The graphs of the reconstructed waveforms are included only to demonstrate the accuracy of the Fourier coefficients used to create the spectral analysis. The reconstructed graphs are not intended to be carbon copies of the original graphs. These graphs are typically slightly smoother and often seem to be moving "forward" at a different rate than the original graphs. The reconstructed graphs are smoother because an exact reconstruction would require an infinite number of upper partials.[80] The reconstructed graphs given here use only a finite number of harmonics, which creates a graph sufficiently close to the original for my purposes. The apparent change on the time scale is due to the fact that the time factor is not accounted for in the reconstructed graphs. Thus, when the original music has note values longer than one subdivided note, the original graph will appear to be delayed slightly behind the reconstructed graph. However, the basic shape of the graph is very clear and easily demonstrates the accuracy of the Fourier coefficients.

The concept of creating a scaling (fractal) structure by using contrapuntal devices (inversion, augmentation, diminution, retrograde) to transform a musical motive is inherent in the structure of all the pieces studied, since each is generated from a single motive. Since this concept is built into the structure of each piece, I have chosen not to include the graphs that show the entrances of the motives.

---

[80] Ibid., 222.

I measured the dimension of each piece using both the method proposed by the Hsus, based on percentage occurrence of the intervals, and the method proposed by Madden, based on the box-counting dimension. In their article, the Hsus do not include all of the intervals when they are calculating the dimensions of the pieces. Although they create graphs that show the logs of the percentage frequencies for all of the pitches, for some unexplained reason, they do not include all the interval classes when creating their trendline and calculating their dimension. Since they give no reasons for excluding some of the intervals, and I have been unable to determine why they excluded some of the intervals, I have chosen to modify their process slightly in my own research. The interval 0 (unison) is excluded in my graphs, like it was in their graphs, because these graphs are based on logarithms, and therefore 0 cannot be used.[81] I have also chosen to exclude any intervals that are larger than an octave. Since intervals above an octave are not common in these pieces by Bach, their small percentage occurrence has a significant effect on the dimension while the intervals themselves are not an important structural element. For the Two-Part Invention in C major (the only piece included in the Hsus' article that I also analyzed), these changes to their method caused the dimension of the music to drop significantly (from 2.4184 to 1.6012). Since I have not studied or analyzed any of the other pieces included in their article, I do not know whether a significant drop in the dimension would always be the result modifying their method in this fashion.

Using this method, the dimensions varied from 0.8862 to 1.8209 for the Two-Part Inventions, and the average was approximately 1.3925. For eleven of the Two-Part Inventions, the Hsus' dimension was higher than Madden's. The Two-Part Inventions in D minor, F major,

[81] Madden's method also does not include the unison intervals since his formula for box-counting requires summing the differences between the pitches, which will not be changed by a 0. Although these intervals can easily be added to the final sum, I have chosen not to add these intervals for two reasons: the unisons are typically not an important structural interval, and I wanted to try to keep some consistency in executing the two methods.

A minor, and Bb major had a smaller dimension when measured with this method than with Madden's.

The dimensions varied from 1.1982 to 2.0051 in the Three-Part Inventions, and the average dimension was approximately 1.6146. Using the modified version of the Hsus' method, all of the dimensions for the Three-Part Inventions were higher than the dimensions measured using Madden's method. Overall, the dimensions for the Three-Part Inventions were typically higher than those of the Two-Part Inventions as compared to Madden's method, which remained consistent throughout all the pieces studied. In the Three-Part Inventions, it consistently appears that the motives that are of a scalar nature or have scalar submotives resulted in higher dimensions; however, this theory does not hold true in the Two-Part Inventions.

Using Madden's method, all the dimensions were between 1 and 2 for the Two-Part and Three-Part Inventions. The right hand dimensions averaged to approximately 1.1977, and the left hand dimensions averaged to approximately 1.2575. For both voices combined, the dimensions varied from 1.1509 to 1.2488. For all but three of the Two-Part Inventions (F major, F minor, A major), the left hand dimension is greater than that of the right hand, although the dimensions are never far apart.

In the Three-Part Inventions, the lowest voice dimension is always the largest, although the dimensions are always relatively close. In the C major, E major, F minor, G minor, A major, and B minor Three-Part Inventions, the upper voice dimension is greater than the middle voice dimension. In the remaining Three-Part Inventions, the middle voice dimension is greater than that of the upper voice. The upper voice dimensions averaged to approximately 1.1797, the middle voice dimensions to approximately 1.1826, and the lowest voice dimensions to approximately 1.2240. The dimensions for all of the voices combined varied from 1.1145 to

1.2953. According to the examples in Madden's book, the low dimension is typical of "smooth" melodies—melodies that are not highly angular or "random." This result is typical for music from the Baroque (1600—1750) period.

The consistency of the results obtained from using Madden's method compared to the widespread and somewhat erratic results from the Hsus' method, along with the practice of excluding some of the intervals, raises questions as to the validity of the Hsus' method. I have still chosen to include the graphs and the measured dimensions despite these questions.

The study of attractors (strange or otherwise) in this music was somewhat futile. Due to the compositional technique used to create these pieces, the music is bound to repeat at some point as the motive is used over and over throughout the piece, meaning that none of these pieces can be strange attractors. However, many of the graphs did have places within the graph that acted as a basin of attraction,[82] indicated by a clustering of notes in a small area. For many of these basins of attraction, although unusual activity or clustering of notes was clear, I was unable to determine precisely which note acted as the attractor, so in the individual summaries, I simply make a note of the fact that there are basins of attraction without specifying which individual note acts as the attractor. For the notes that I was able to identify, the basins of attraction usually occur at the tonic and dominant pitches for the piece. All the graphs showed correlation along a diagonal line, indicating a strong sense of tonality, which is to be expected in music of the time period.

The Fourier analysis and resulting spectral graphs also proved to be fairly consistent throughout the set of pieces, although there were a few with unusual qualities (see Two-Part Inventions in E major, F minor, and Bb major and Three-Part Inventions in D minor, Eb major, F

[82] A basin of attraction is the set of points, or notes, within which any given point will eventually move toward a specific attractor.

minor, Bb major, and B minor). None of the graphs had an even distribution throughout the harmonics, which would indicate a highly angular and random curve, although the Three-Part Invention in Bb major has a very interesting distribution and comes the closest to having an even distribution. Typically, the graphs begin with several high or moderately high amplitudes, and eventually, typically by the fifteenth harmonic, the amplitudes will gradually drop off and become stable, and they will remain low and relatively stable throughout the remainder of the spectrum. A few of the graphs have energy in the upper harmonics, indicating a slightly more complex sound curve.

As expected from the nature of the pieces I chose to analyze, my findings were similar for all of the pieces. Since all the pieces were built on the same compositional technique, all the pieces have the same basic features, and thus many of these structural analyses had similar results. The fact that all of the pieces are tonal and that all are built on imitative counterpoint, also meant that most of the results would be similar or the same for all the pieces I chose to study.

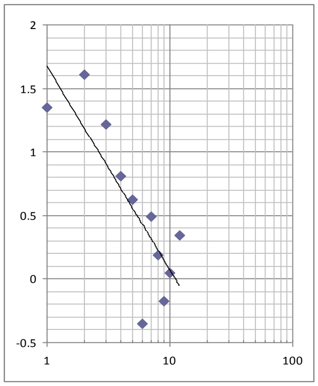
Two-Part Invention in C Major, No. 1, BWV 772 (1723)

Dimension, Madden

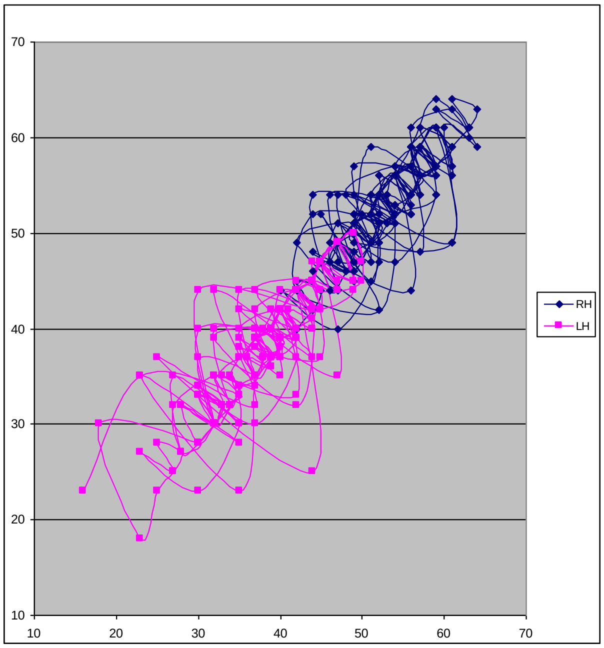
- Right hand: 1.1802
- Left hand: 1.2086

Dimension, Hsu

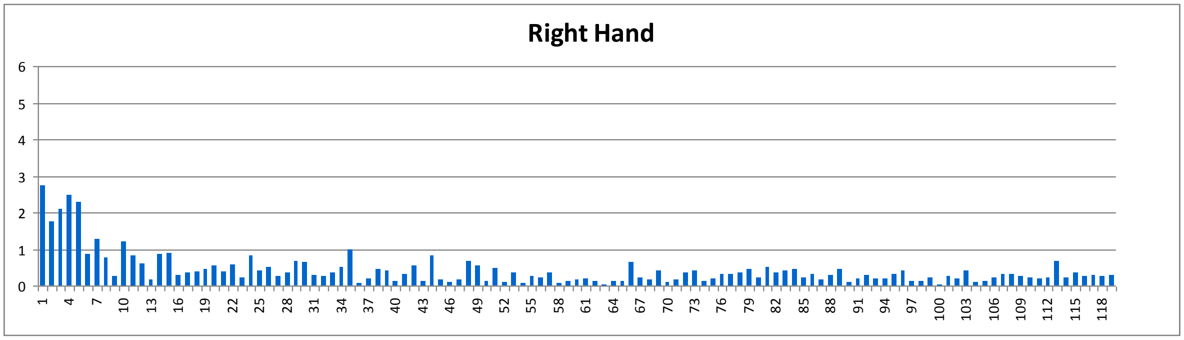
- Combined dimension: 1.6012

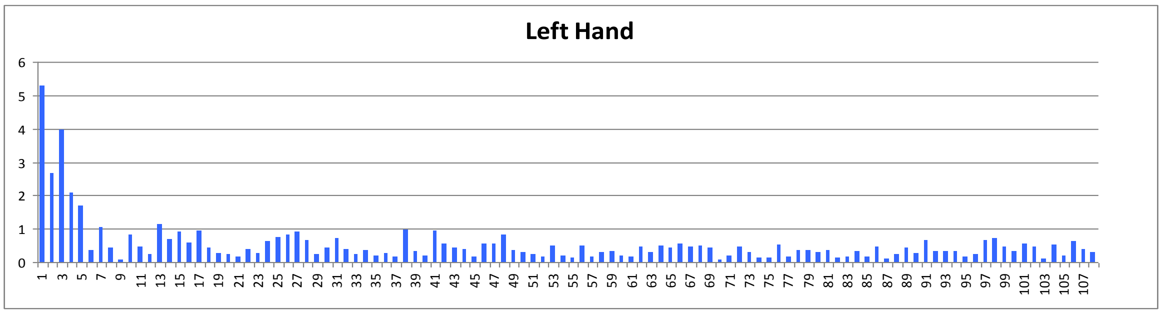


Attractor Plot

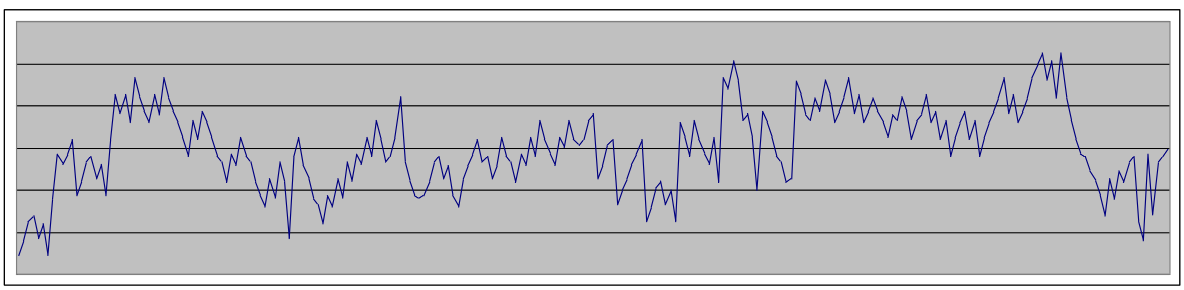


Spectral Analysis

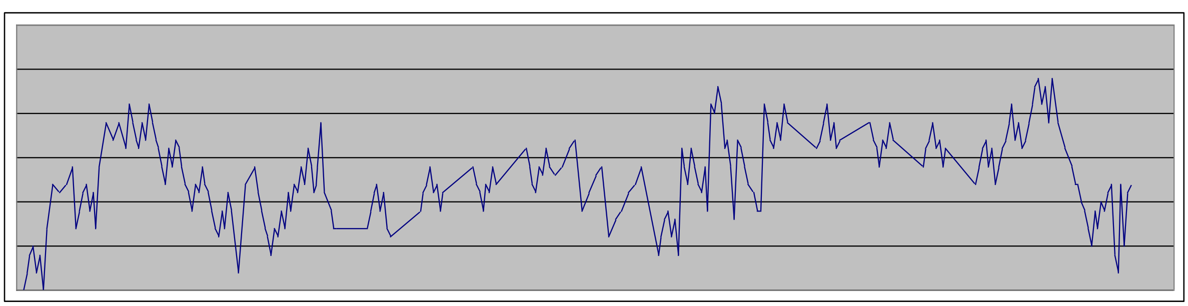




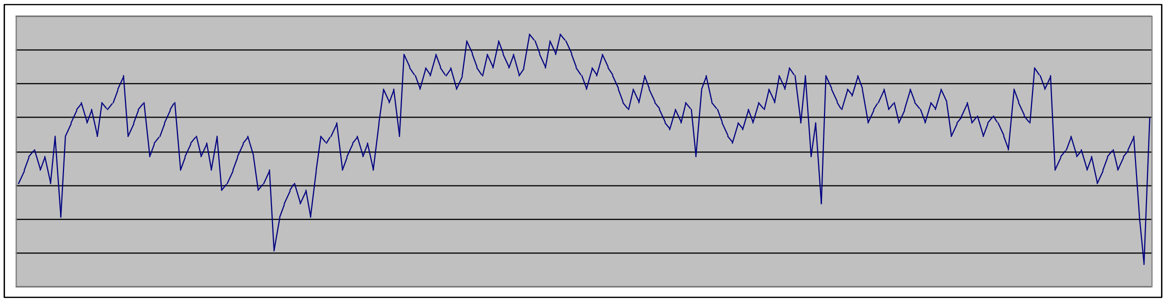
Inverse Transforms



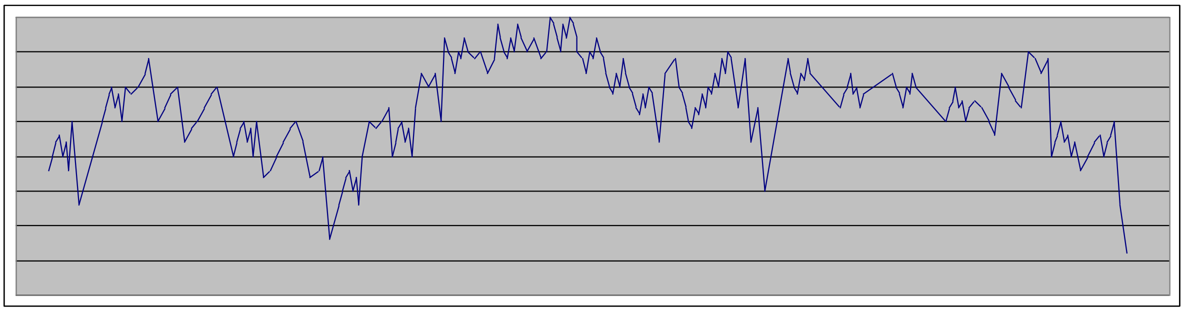
Inverse Transform (Right Hand)



Original Graph (Right Hand)



Inverse Transform (Left Hand)



Original Graph (Left Hand)

The dimensions for the right and left hands are quite close. The fact that the left hand has a slightly larger dimension indicates that the left hand line is probably slightly more angular, which is confirmed by looking at the music and considering the character of the left hand line, especially at cadential moments when the left hand consistently makes octave jumps.

The attractor graph is clearly correlated along the diagonal line, although the left hand shows slightly more deviation in its orbits from this center (due to the octave intervals) than the right hand. There are two strong basins of attraction as well as some weaker basins of attraction.

The right hand harmonic spectrum is dominated by the first five harmonics, which all have approximately the same size amplitudes. While there is some energy in the upper harmonics, there is no significant activity.

The left hand harmonic spectrum is also dominated by the first five harmonics, but primarily by the first and third harmonics, which have markedly higher amplitudes than the others. There is no significant energy in the upper harmonics.

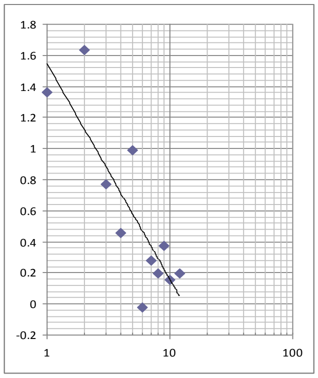
Two-Part Invention in C Minor, No. 2, BWV 773 (1723)

Dimension, Madden

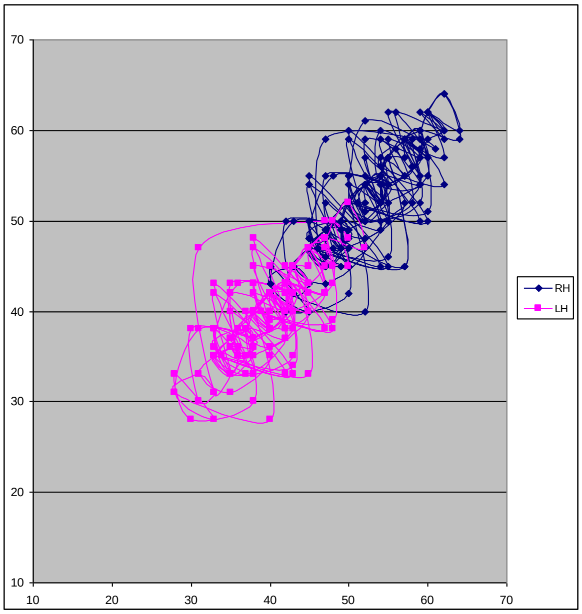
- Right hand: 1.1739
- Left hand: 1.1802

Dimension, Hsu

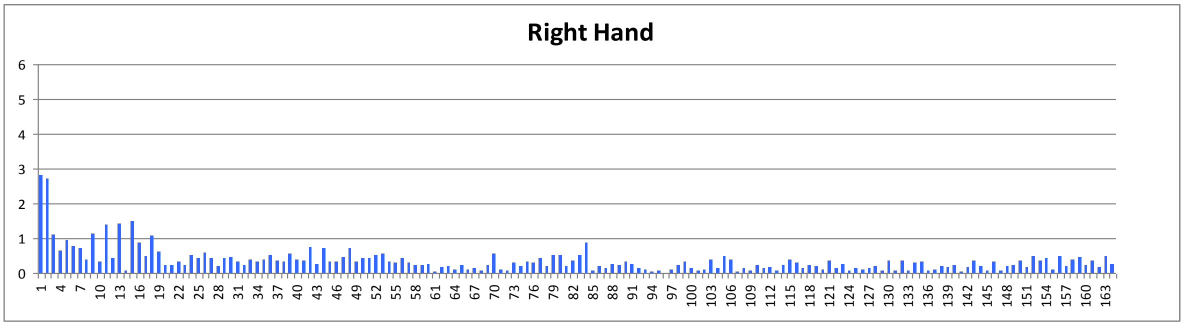
- Combined dimension: 1.3885

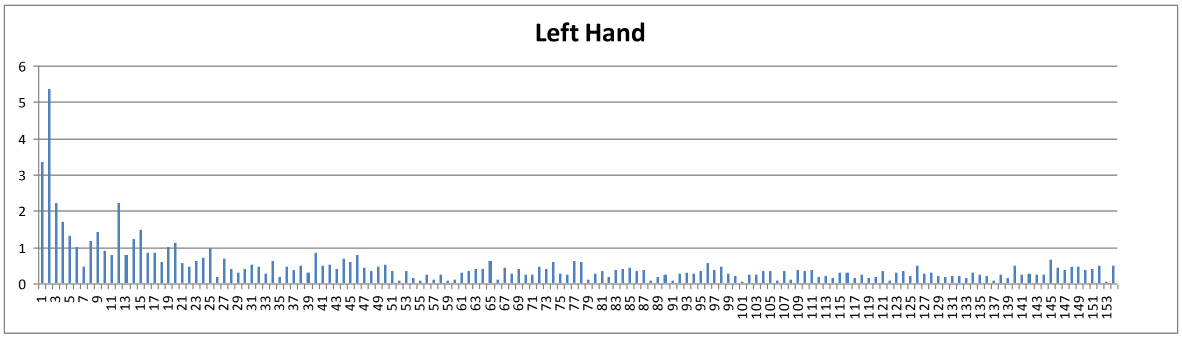


Attractor Plot

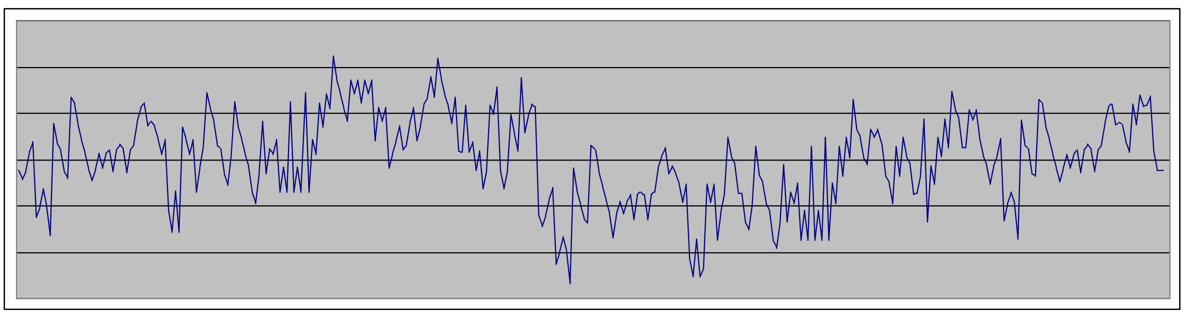


Spectral Analysis

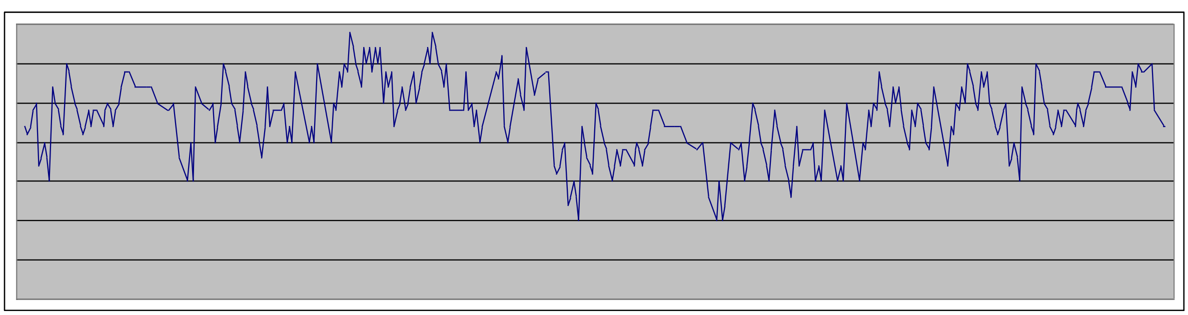




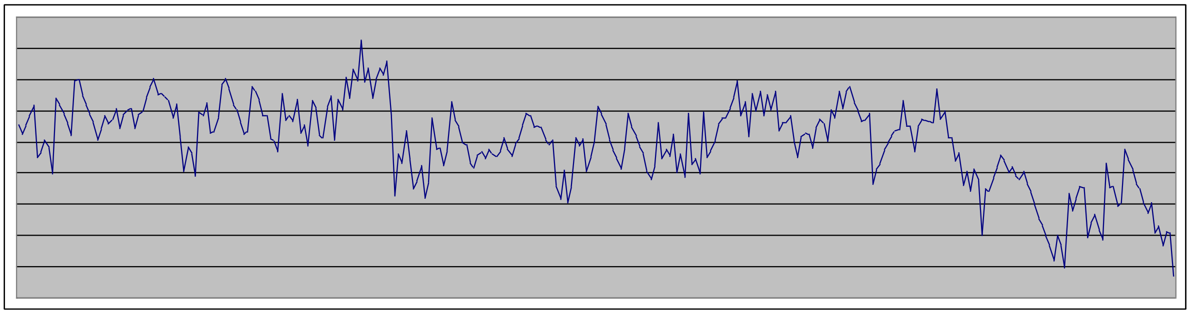
Inverse Transforms



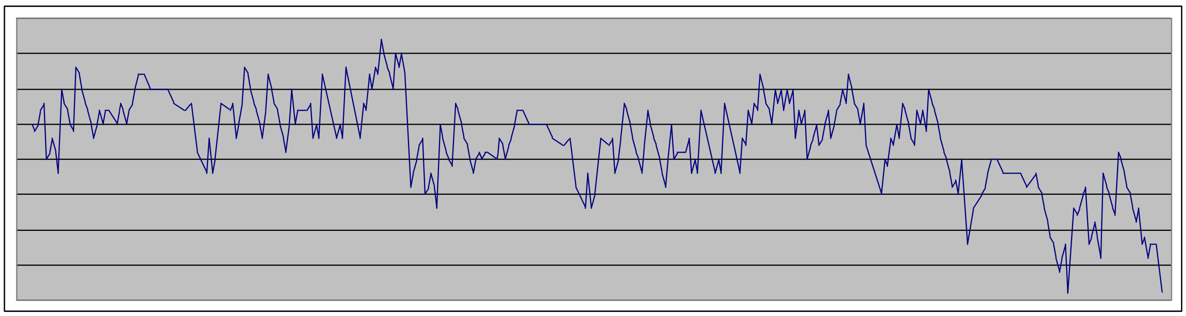
Inverse Transform (Right Hand)



Original Graph (Right Hand)



Inverse Transform (Left Hand)



Original Graph (Left Hand)

The dimensions for the right and left hands are again almost identical, only differing by approximately 0.0029. This corresponds to the music, since there are not significantly more large or more frequent leaps in one hand than in the other.

The right hand harmonic spectrum begins with two high amplitude harmonics, which are followed by a slight increase in amplitude for harmonics nine through sixteen, after which there is no significant activity in the higher harmonics.

The left hand harmonic spectrum begins with a harmonic of medium amplitude which is immediately followed by a second harmonic with a higher amplitude. There is some activity through harmonic nineteen, with low amplitudes at harmonics three and twelve, and no significant activity in the upper harmonics.

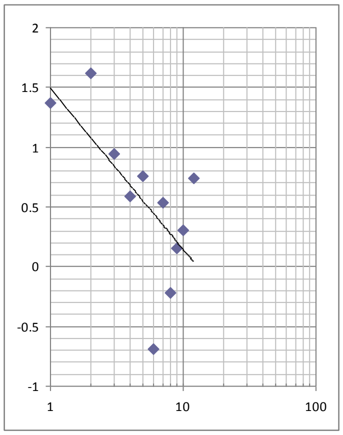
Two-Part Invention in D Major, No. 3, BWV 774 (1723)

Dimension, Madden

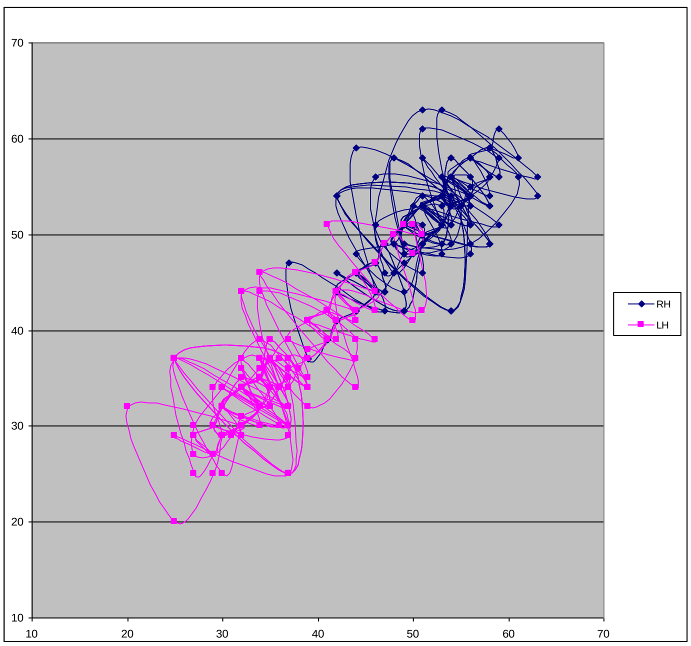
- Right hand: 1.2014
- Left hand: 1.2080

Dimension, Hsu

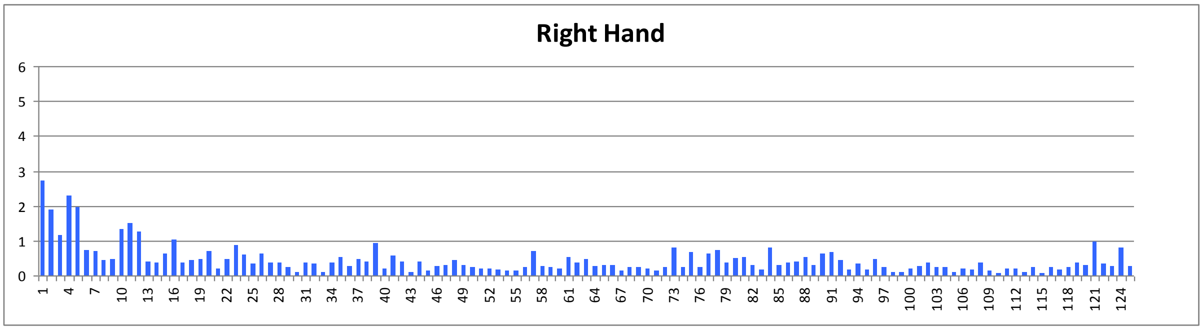
- Combined dimension: 1.3489

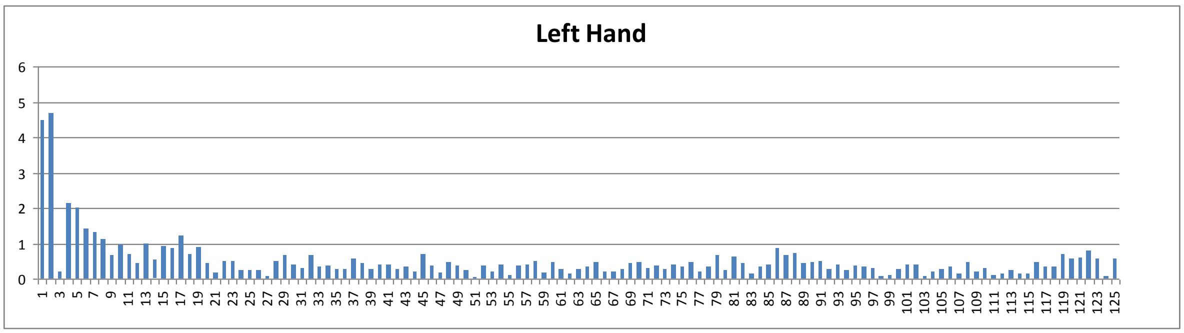


Attractor Plot

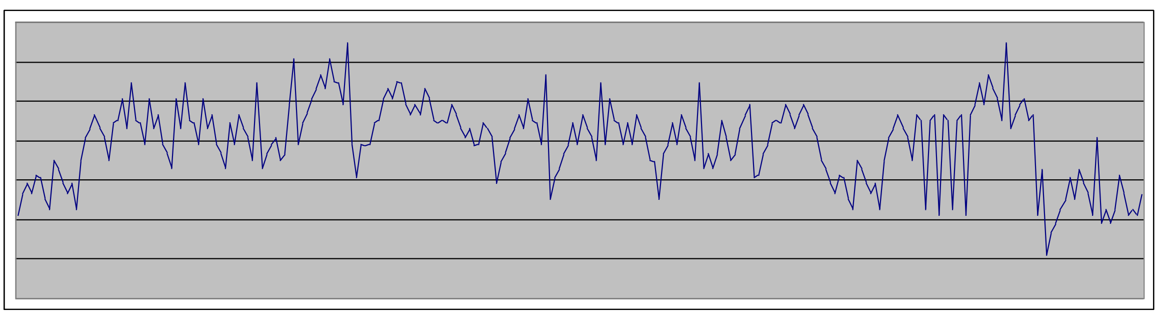


Spectral Analysis

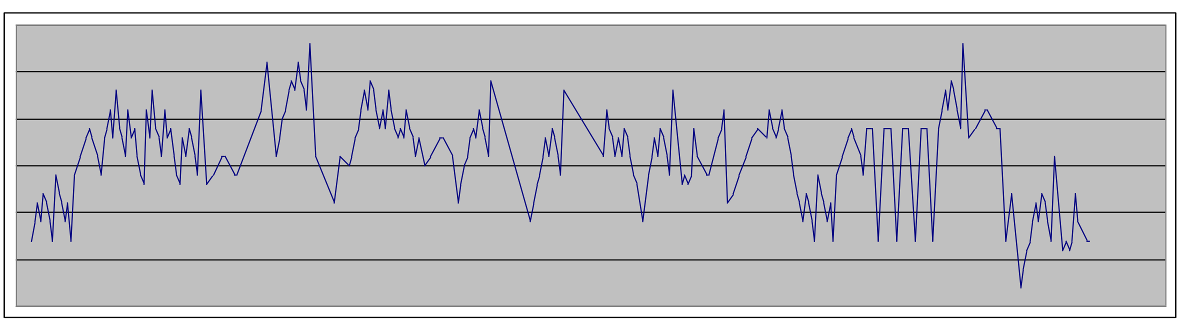




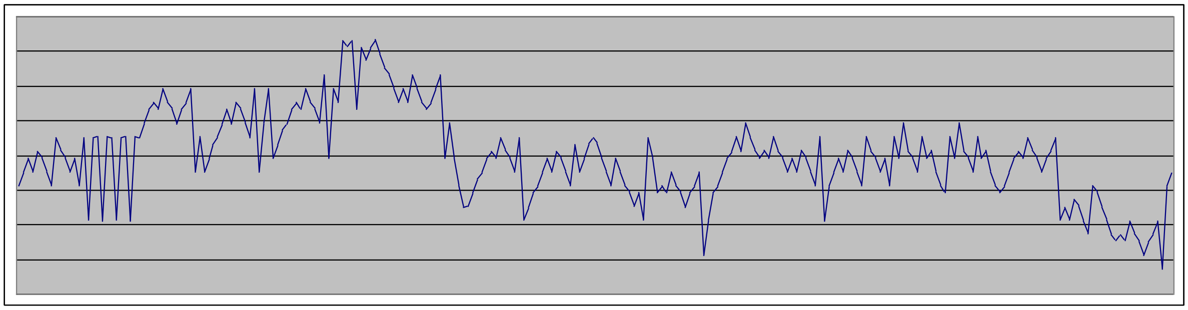
Inverse Transforms



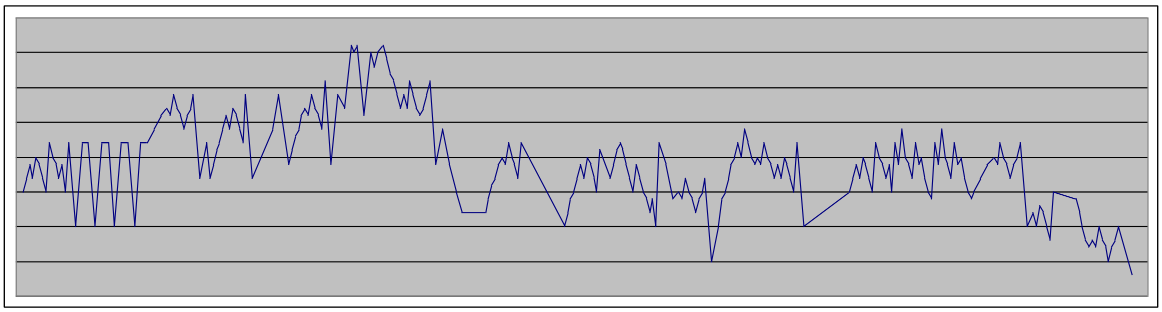
Inverse Transform (Right Hand)



Original Graph (Right Hand)



Inverse Transform (Left Hand)



Original Graph (Left Hand)

The attractor graph is again correlated along the diagonal line, although there are some interesting orbital deviations from the diagonal line in both hands. This is caused by the motivic use of large jumps in both hands, specifically the use of octave leaps—ten octave leaps in the right hand and seventeen octave leaps in the right hand. There are two strong basins of attraction within this graph.

The right hand harmonic spectrum is dominated by the first five harmonics, which have varying amplitudes, but are still significantly higher than the other harmonics. There is no significant activity in the upper harmonics.

The left hand harmonic spectrum is slightly unusual in that it begins with two high amplitude harmonics that are followed by a very small third harmonic, after which harmonics four through seven begin at a medium amplitude and then gradually decrease.

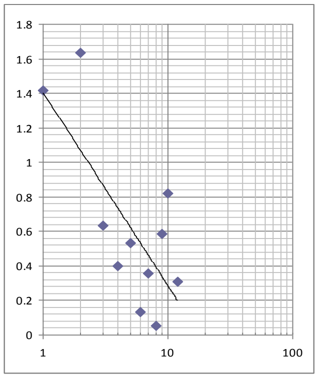
Two-Part Invention in D Minor, No. 4, BWV 775 (1723)

Dimension, Madden

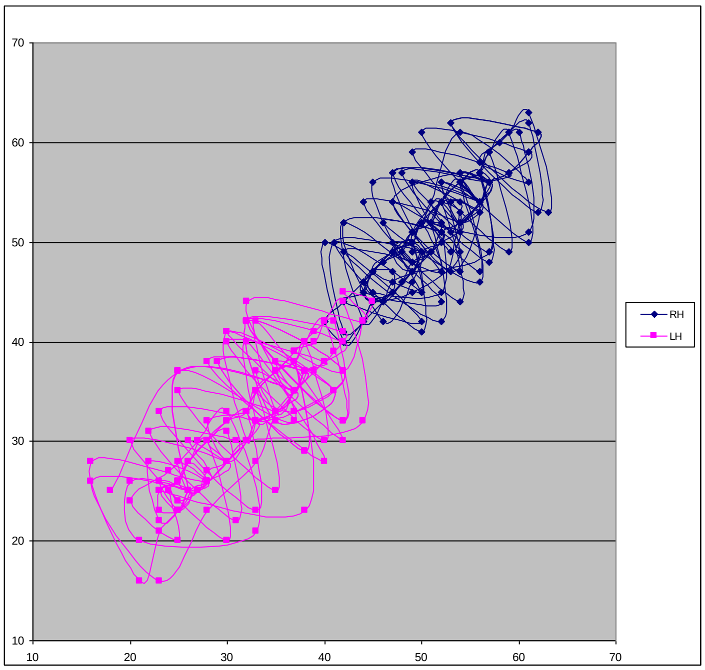
- Right hand: 1.1979
- Left hand: 1.2437

Dimension, Hsu

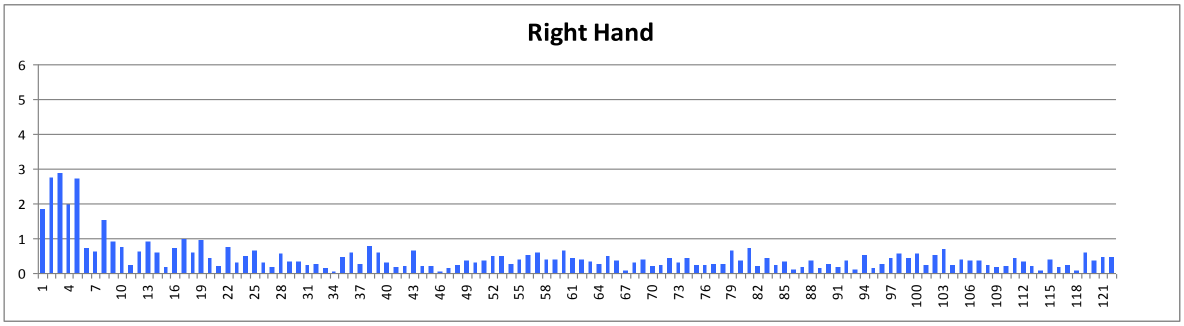
- Combined dimension: 1.1174

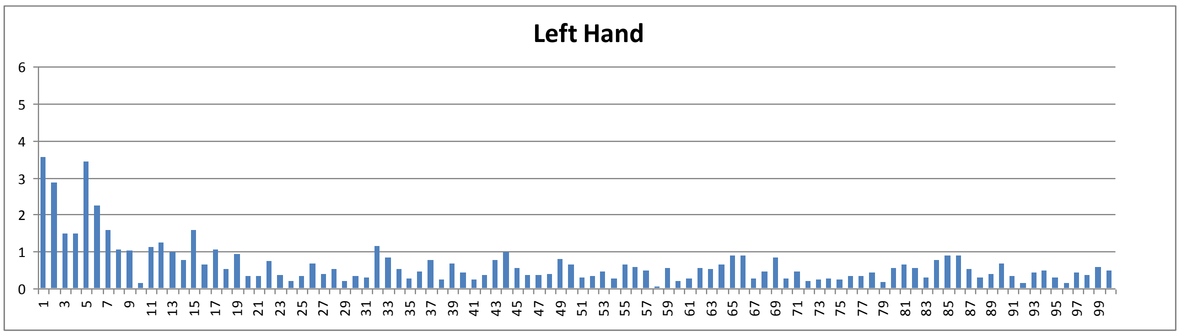


Attractor Plot

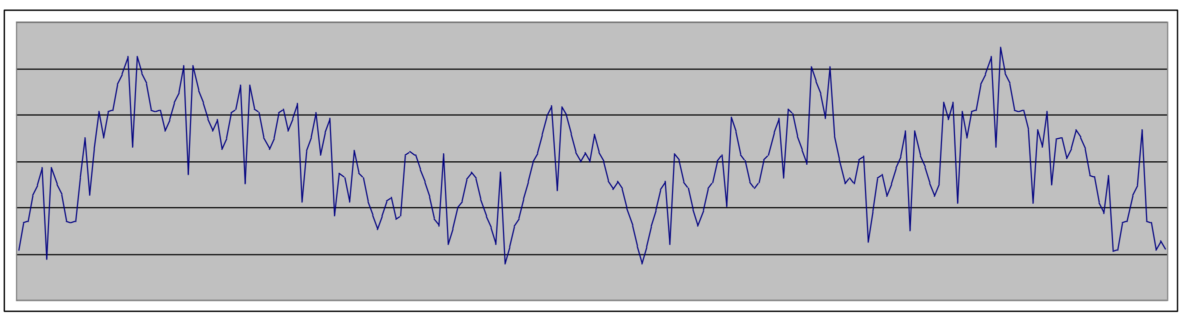


Spectral Analysis

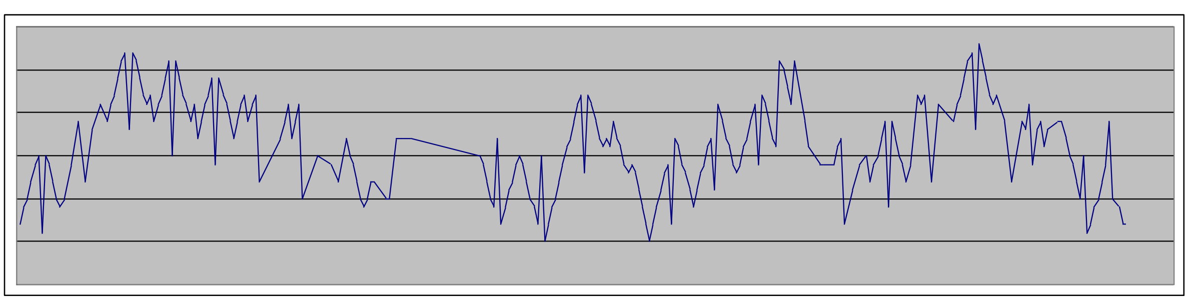




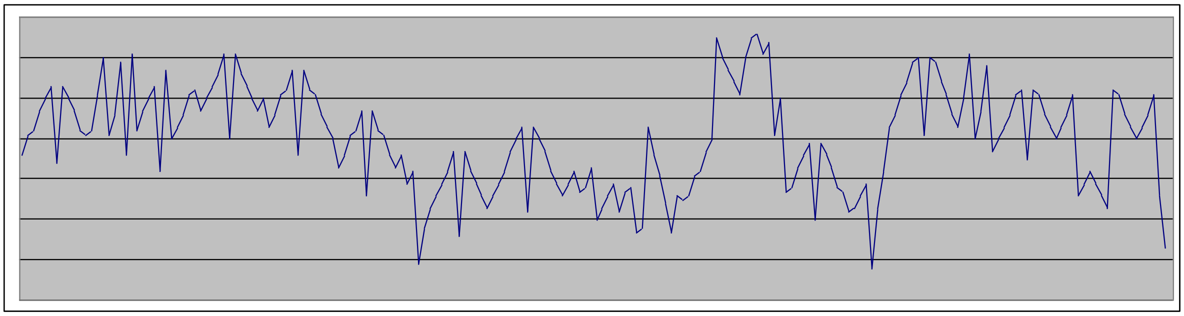
Inverse Transforms



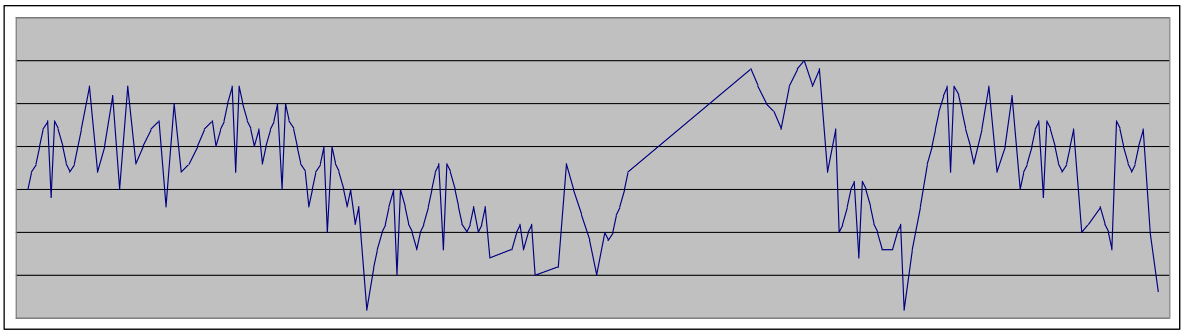
Inverse Transform (Right Hand)



Original Graph (Right Hand)



Inverse Transform (Left Hand)



Original Graph (Left Hand)

The dimensions for each voice are very close. The left hand dimension is only slightly larger, which is due to the slightly more angular nature of the left hand line. It is interesting to note that for this piece the Hsu dimension drops significantly below the Madden dimension. The reasons for this sudden drop are unclear.

The attractor graph does not have any particularly striking behavior. It is worth noting that the left hand seems to be somewhat less strictly correlated than the right hand, since it has looser orbits and the orbits are less consistent. Also, the right hand appears to have at least one strong basin of attraction, while the left hand seems to have weak basins of attraction at best.

The harmonic spectrum for the right hand has highest amplitudes in the first five harmonics, with another high amplitude at the eighth harmonic and no other significant activity in the higher harmonics.

The harmonic spectrum for the left hand begins with two high amplitude harmonics, which are followed by two harmonics of smaller amplitude. There is an interesting spike at the fifth harmonic, after which the spectrum gradually settles down. There is slightly more activity in the upper harmonics than is typical; the amplitudes do not completely settle down until about the thirty-third harmonic.

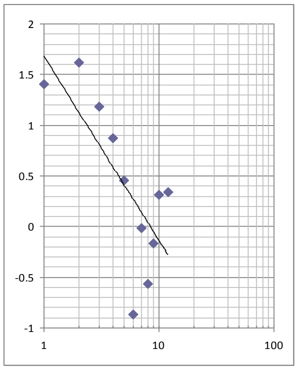
Two-Part Invention in Eb Major, No. 5, BWV 776 (1723)

Dimension, Madden

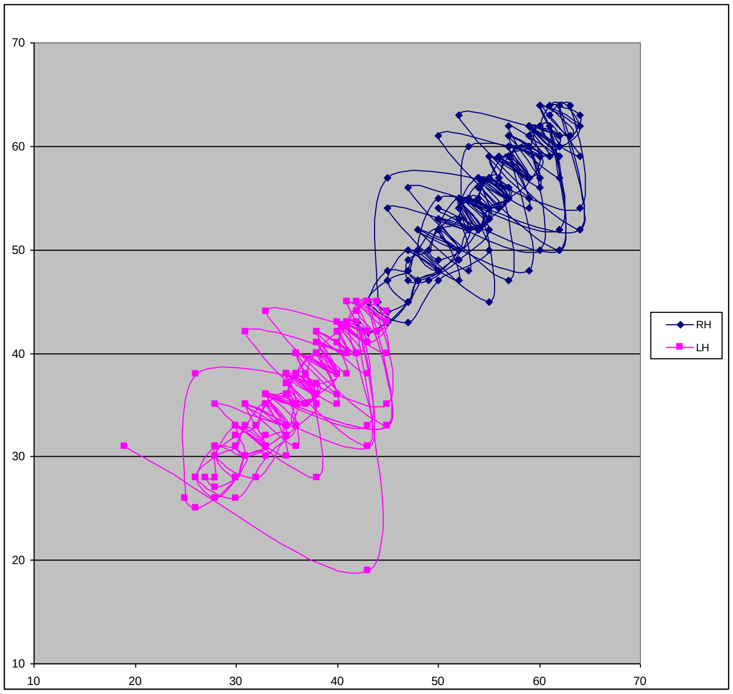
- Right hand: 1.1686
- Left hand: 1.1758

Dimension, Hsu

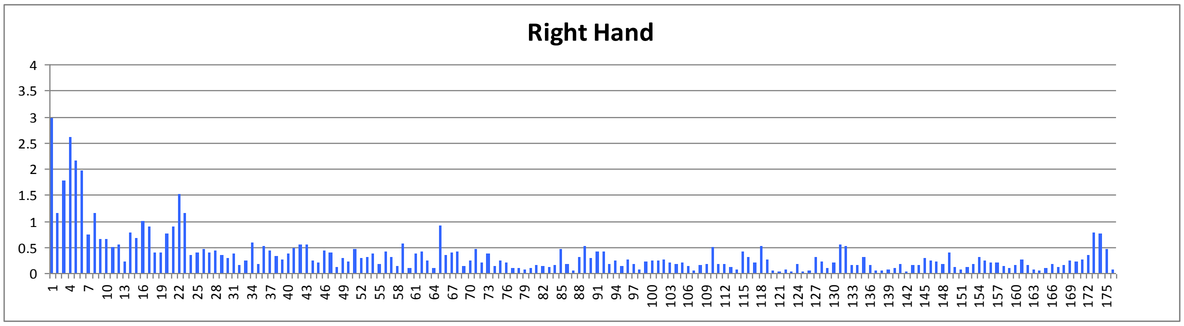
- Combined dimension: 1.8181

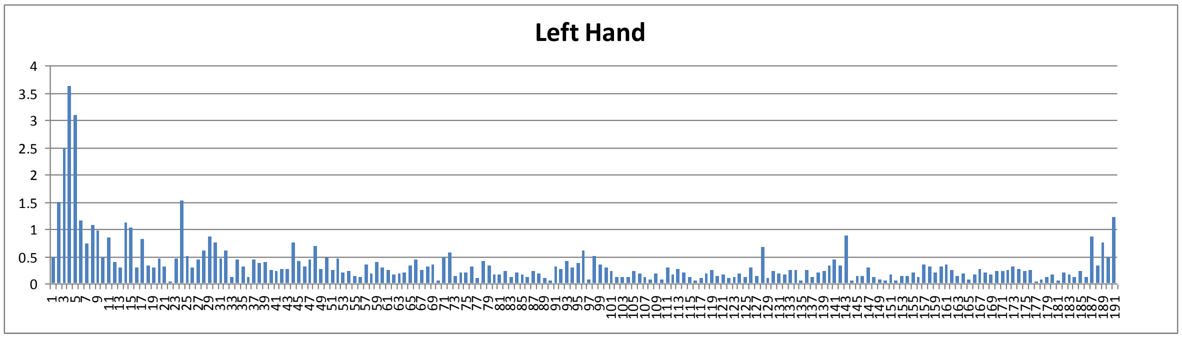


Attractor Plot

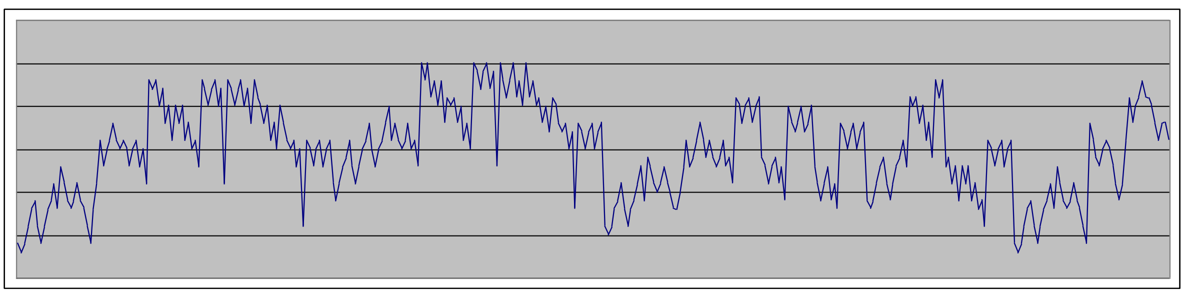


Spectral Analysis

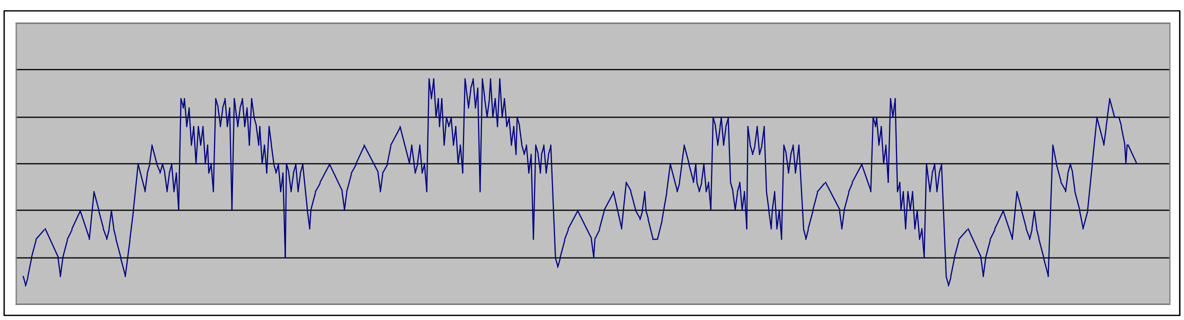




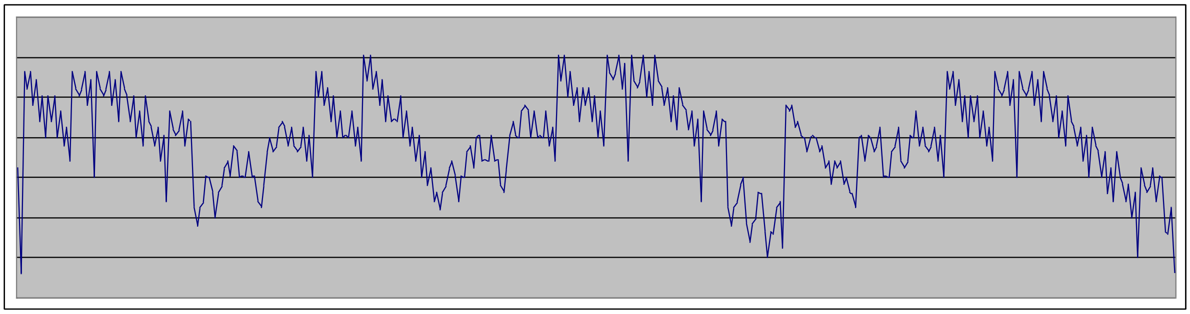
Inverse Transforms



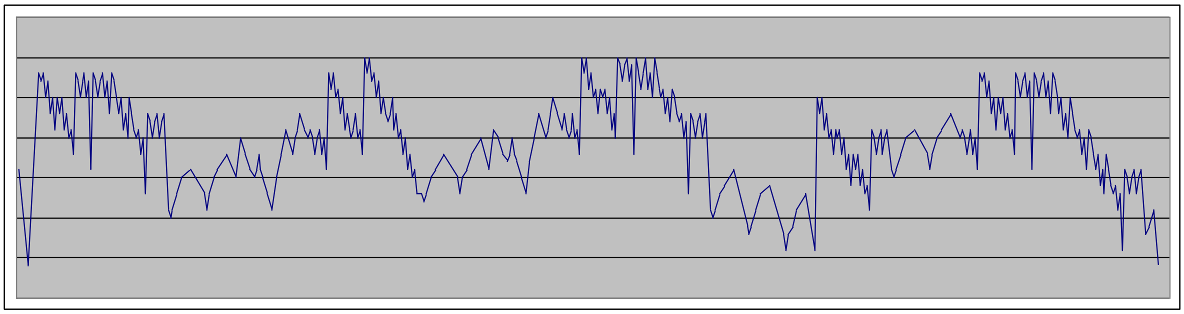
Inverse Transform (Right Hand)



Original Graph (Right Hand)



Inverse Transform (Left Hand)



Original Graph (Left Hand)

The Hsu dimension for this piece is relatively high, especially when compared to Madden’s numbers.

The attractor shows a rather interesting orbit, which is caused by a relatively high number of larger intervals (m7, M7, P8). There are several strong basins of attraction spread throughout the graph.

The harmonic spectrum of the right hand begins with a high amplitude that is followed by a single low amplitude harmonic and then a small group of high amplitude harmonics, after which the harmonics gradually stabilize except for small spikes at about the twenty-first and sixty-fifth harmonics. There is no significant energy in the upper harmonics until the last few harmonics, where there is a small increase in amplitude.

The harmonic spectrum of the left hand begins with a rather low first harmonic after which the amplitudes increase through the fourth harmonic, which has a rather high amplitude. After the fourth harmonic, the amplitudes gradually drop off and stabilize, although there is more energy than usual throughout the upper harmonics, with small spikes at the 128th and 142nd harmonic, and a small group of harmonics with slightly increased amplitude at the end of the spectrum.

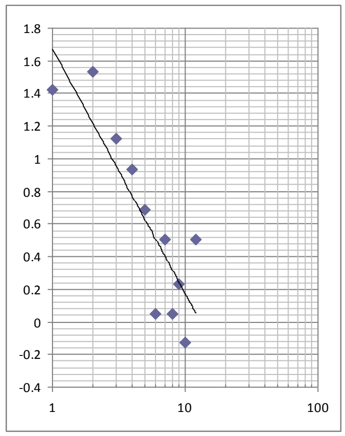
Two-Part Invention in E Major, No. 6, BWV 777 (1723)

Dimension, Madden

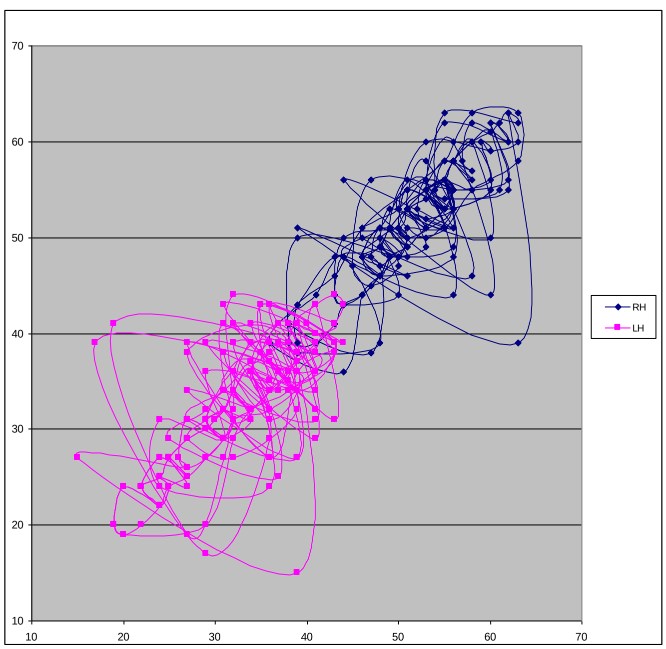
- Right hand: 1.1811
- Left hand: 1.2371

Dimension, Hsu

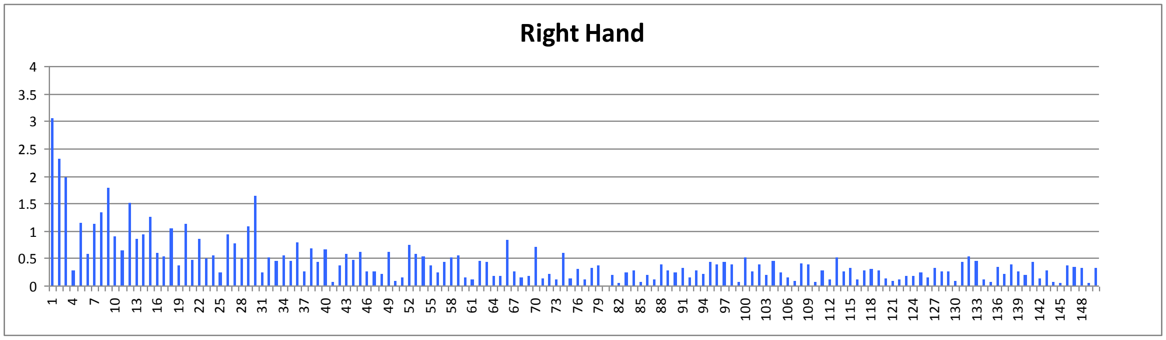
- Combined dimension: 1.5006

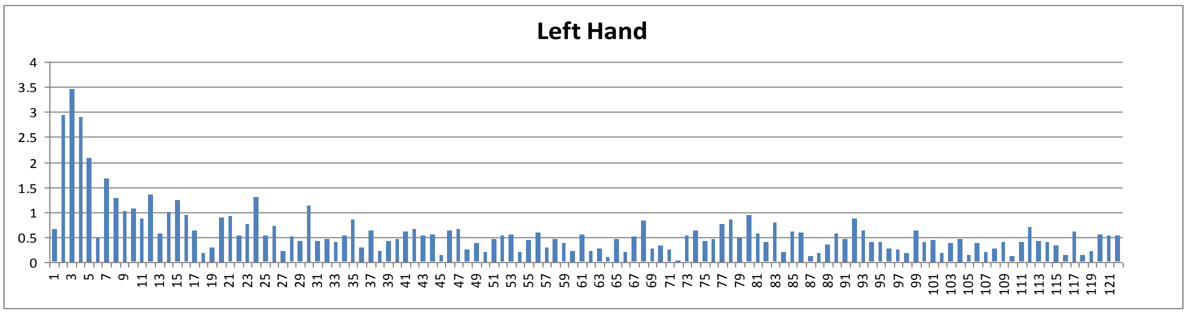


Attractor Plot

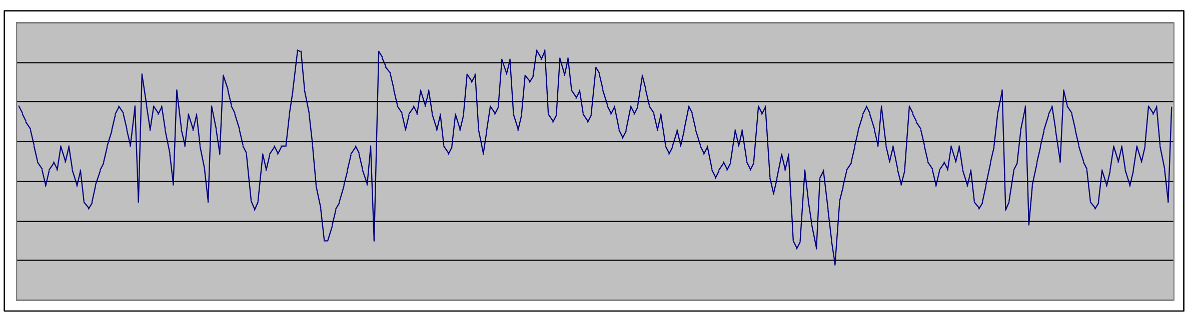


Spectral Analysis

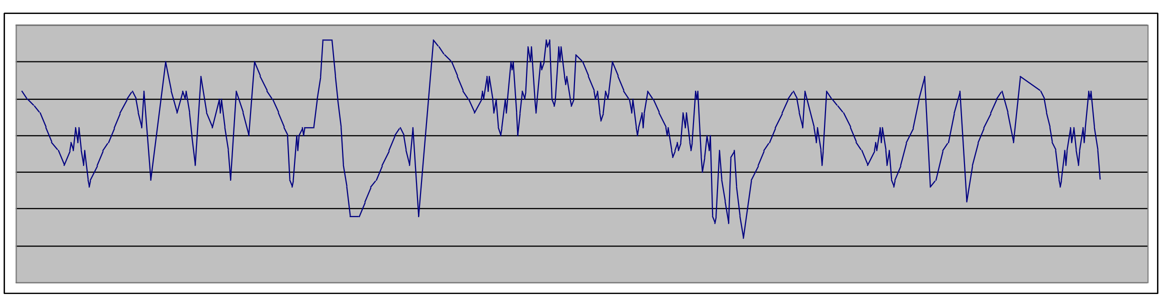




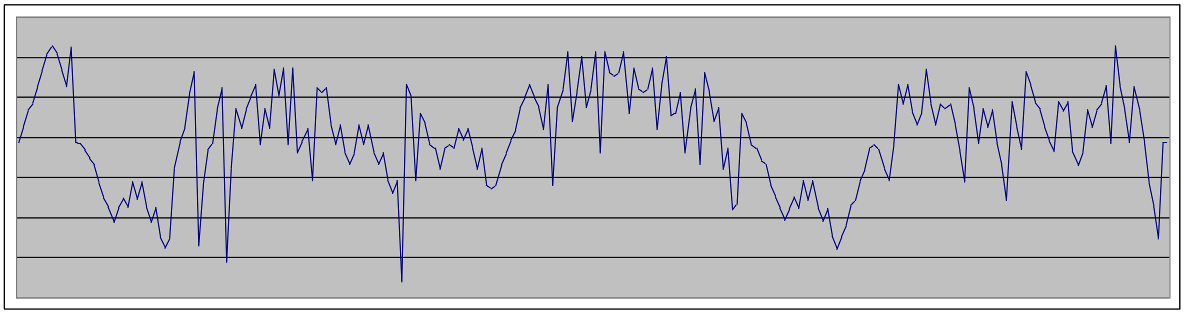
Inverse Transforms



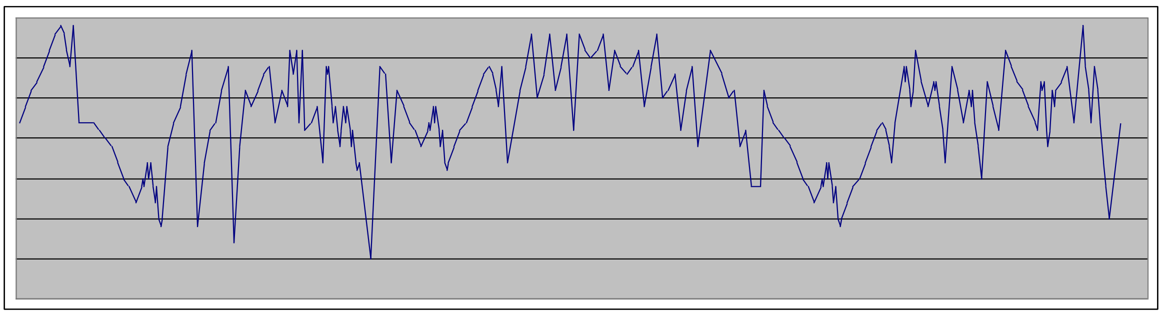
Inverse Transform (Right Hand)



Original Graph (Right Hand)



Inverse Transform (Left Hand)



Original Graph (Left Hand)

The attractor graph shows a strong correlation along the diagonal line, but along with the wide orbital pattern (due to the occurrence of large intervals, both octaves and intervals larger than an octave), there also appears to be an unusual distribution along the diagonal line. There appear to be several different centers (basins) of attraction of varying strength along the diagonal line, since the notes tend to cluster in five relatively distinct areas, two of which are strong, two of which are weaker, and one of which (at the lowest end of the left hand) is the weakest, having only a few notes clustering around it. The piece as a whole does not represent a strange attractor, since parts of it repeat, but there are definite indications of attractors within the music.

The harmonic spectrum of the right hand is somewhat unusual with no strong dominating harmonics after the first harmonic, which has a high amplitude. However, the amplitudes do not completely settle down until relatively late, at about the seventy-fifth harmonic.

The harmonic spectrum of the left hand has even more energy in the upper harmonics. Although there are no significant spikes in amplitude after the first five harmonics, the energy continues all through the upper harmonics.

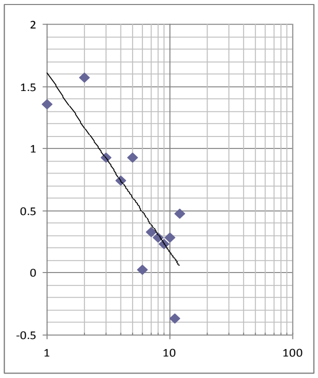
Two-Part Invention in E Minor, No. 7, BWV 778 (1723)

Dimension, Madden

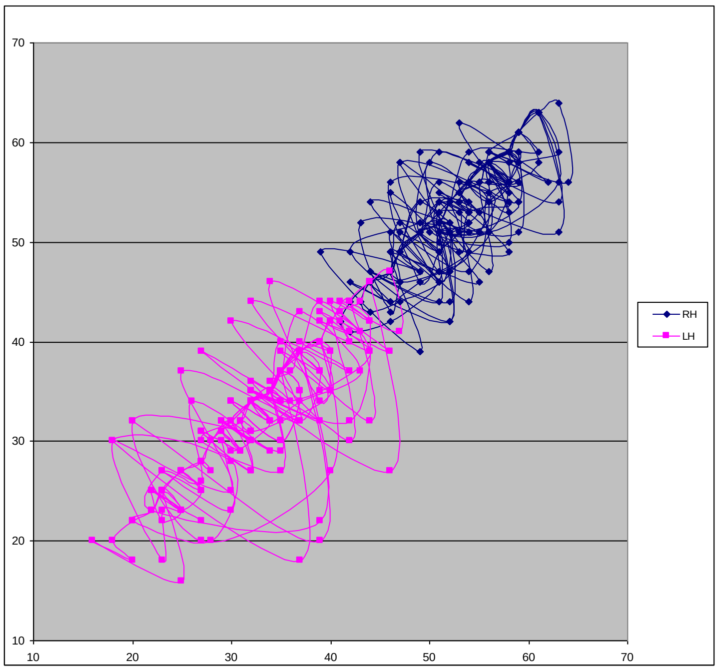
- Right hand: 1.1928
- Left hand: 1.2290

Dimension, Hsu

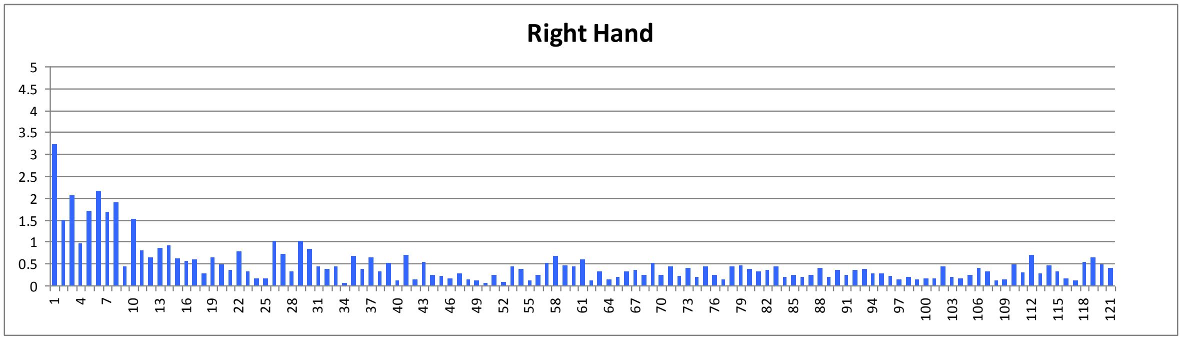
- Combined dimension: 1.4426

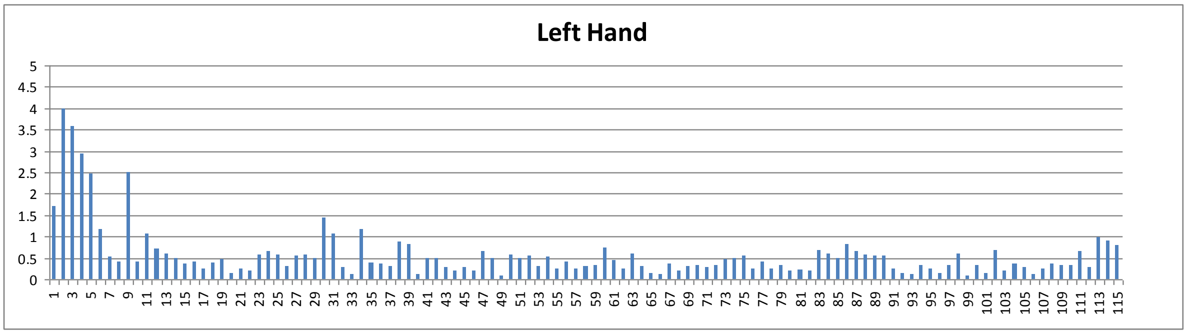


Attractor Plot

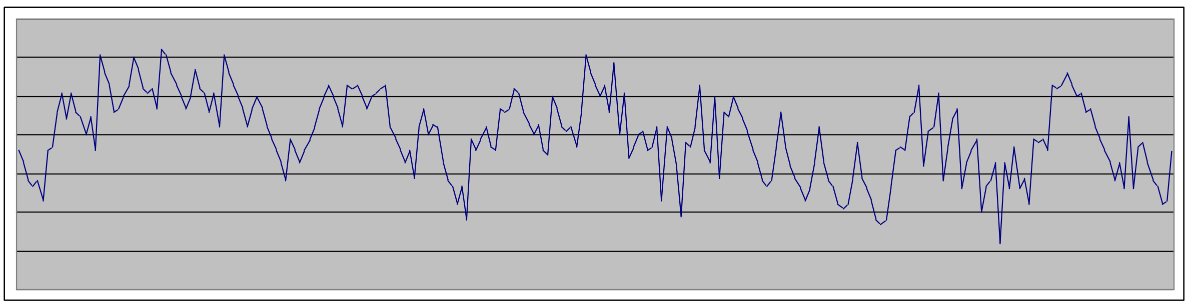


Spectral Analysis

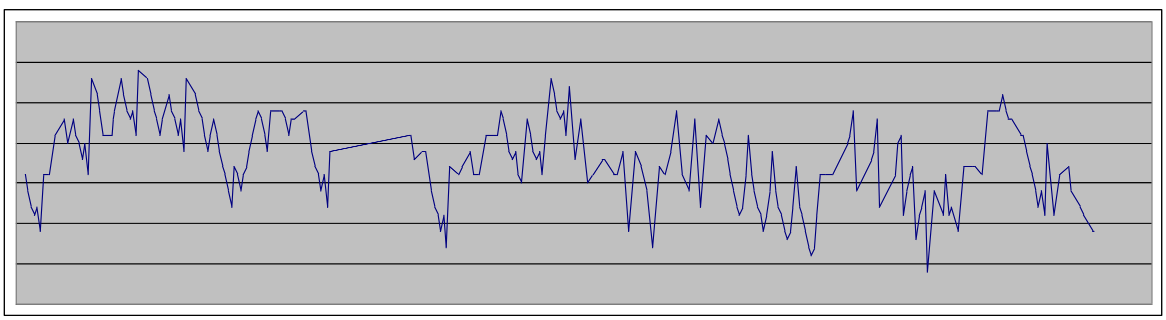




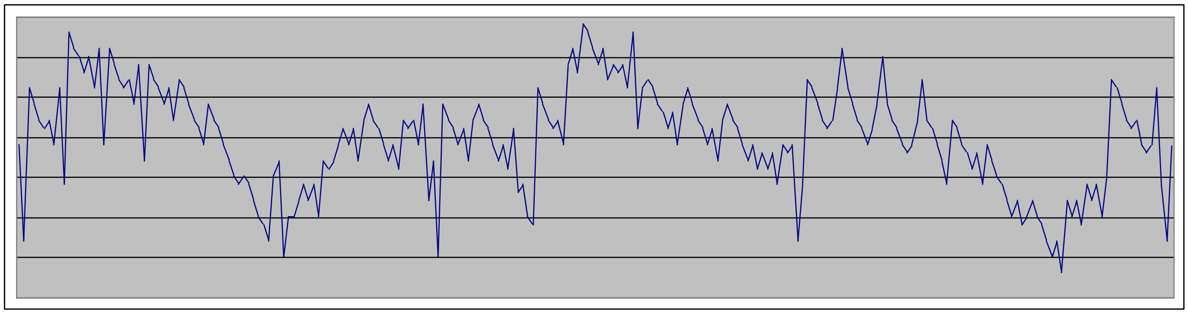
Inverse Transforms



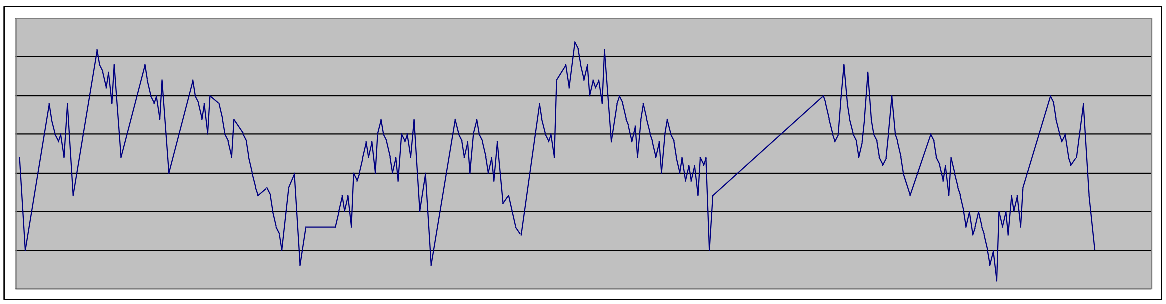
Inverse Transform (Right Hand)



Original Graph (Right Hand)



Inverse Transform (Left Hand)



Original Graph (Left Hand)

The left and right hand dimensions are close. The slightly larger left hand dimension is due to the number of octave intervals in the left hand line (13) as well as the five intervals in the left hand line that are larger than an octave.

The attractor plot does not show any unusual behavior. The wide orbits in both hands are due to the occurrence of large intervals. There is one strong and one weak basin of attraction in the right hand, and a weak basin of attraction in the left hand.

The harmonic spectrum of the right hand has high amplitude through the first ten harmonics, after which the amplitudes gradually drop off. Although there are a few more slight rises in amplitude (at harmonics 14, 30, 58, and 112), there is not a significant amount of energy in the higher amplitudes.

The harmonic spectrum of the left hand begins with a mid-amplitude harmonic followed by high amplitude harmonics through the fifth harmonic, after which the amplitudes gradually drop off except for a spike at the ninth harmonic. Although the amplitudes drop after the ninth harmonic, there is still some amount of energy in the upper harmonics, but without any significant increase in the amplitudes.

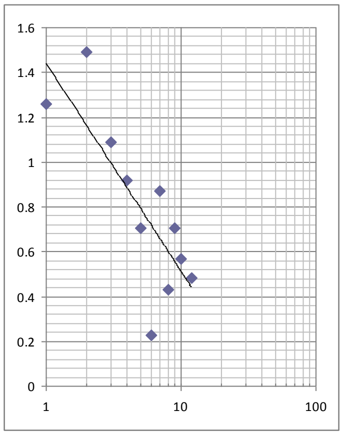
Two-Part Invention in F Major, No. 8, BWV 779 (1723)

Dimension, Madden

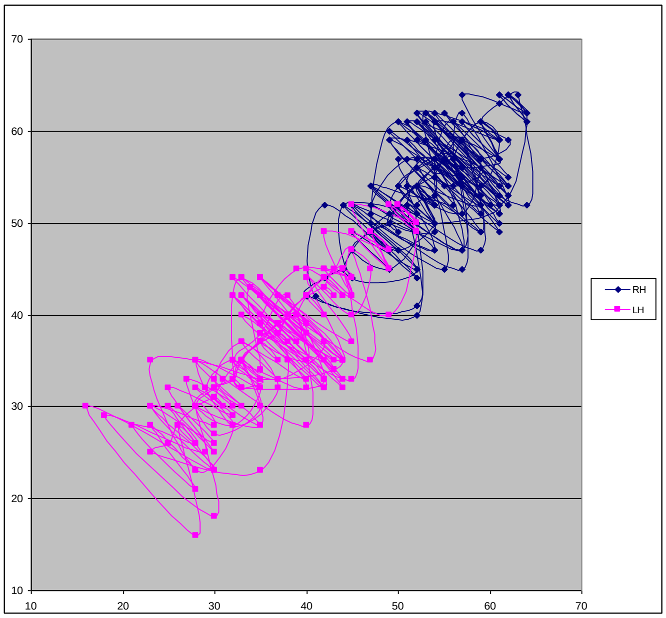
- Right hand: 1.2467
- Left hand: 1.2373

Dimension, Hsu

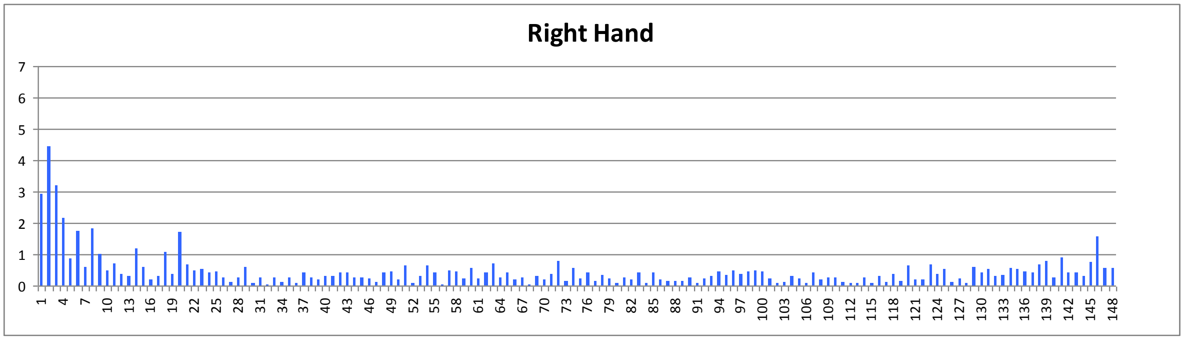
- Combined dimension: 0.9277

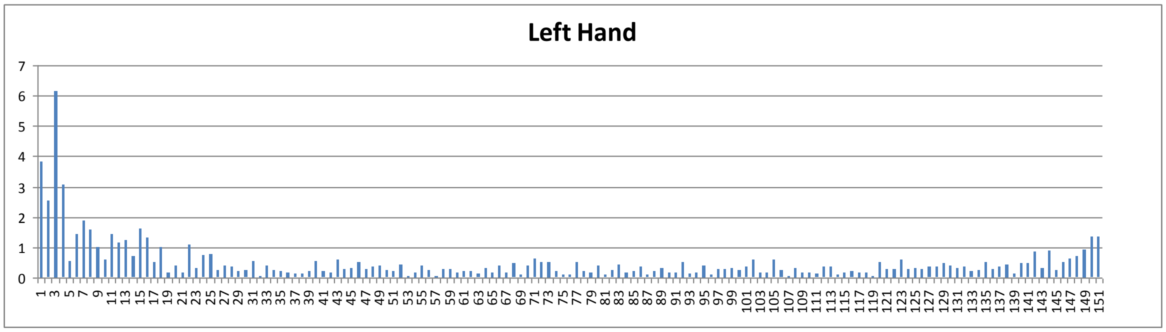


Attractor Plot

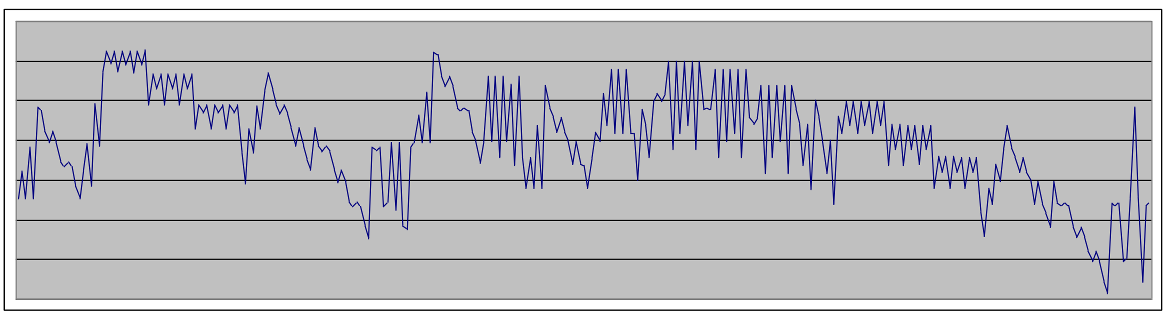


Spectral Analysis

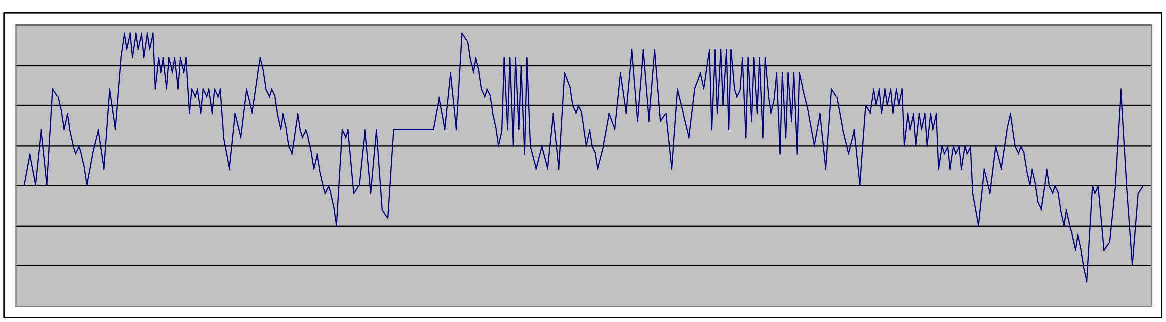




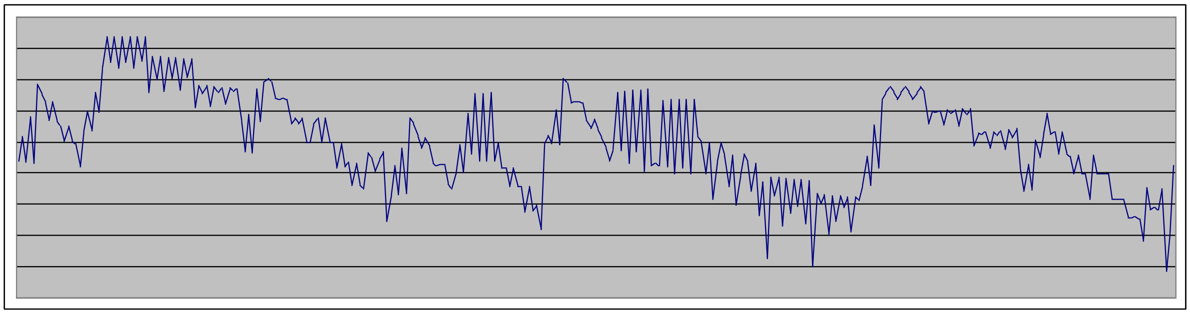
Inverse Transforms



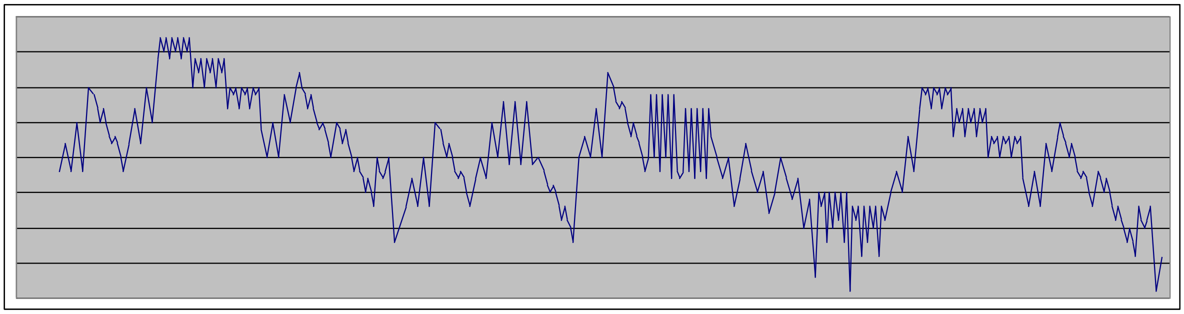
Inverse Transform (Right Hand)



Original Graph (Right Hand)



Inverse Transform (Left Hand)



Original Graph (Left Hand)

The two dimensions are almost identical for both hands in this invention.

While the attractor graph is once again correlated along the diagonal line, there are some unusual patterns in this graph due to the Alberti-like figure used throughout the piece, causing the orbits to rock back and forth in straight lines rather than in circular orbits. This graph also shows two distinctly strong basins of attraction along the diagonal line and several other attractors of varying strengths.

The two harmonic spectrums look very similar for this invention. Both begin with high amplitude harmonics, and the energy in the harmonics gradually drops off until about the twentieth harmonic. Both also have an interesting, although rather small, spike close to the end of the spectrum.

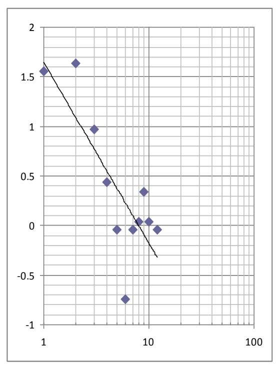
Two-Part Invention in F Minor, No. 9, BWV 780 (1723)

Dimension, Madden

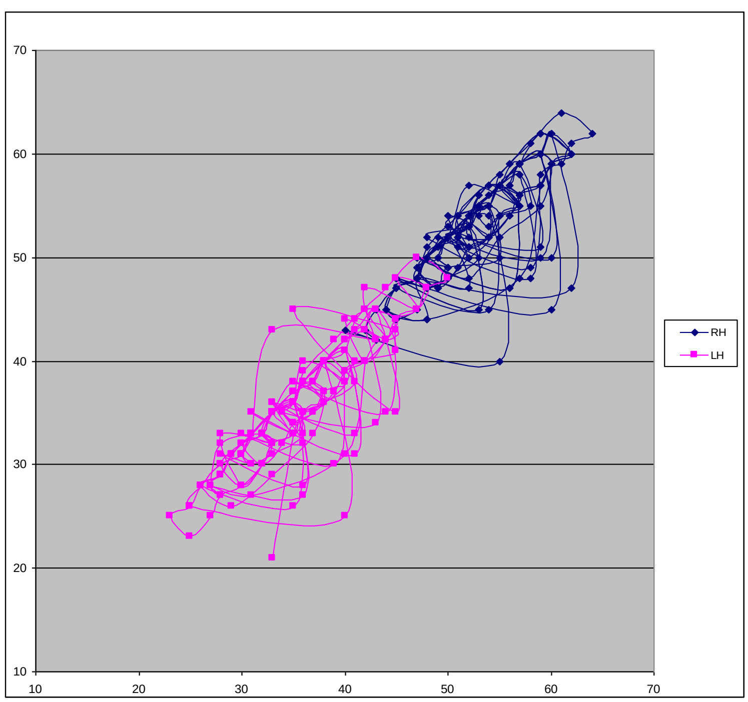
- Right hand: 1.1510
- Left hand: 1.1509

Dimension, Hsu

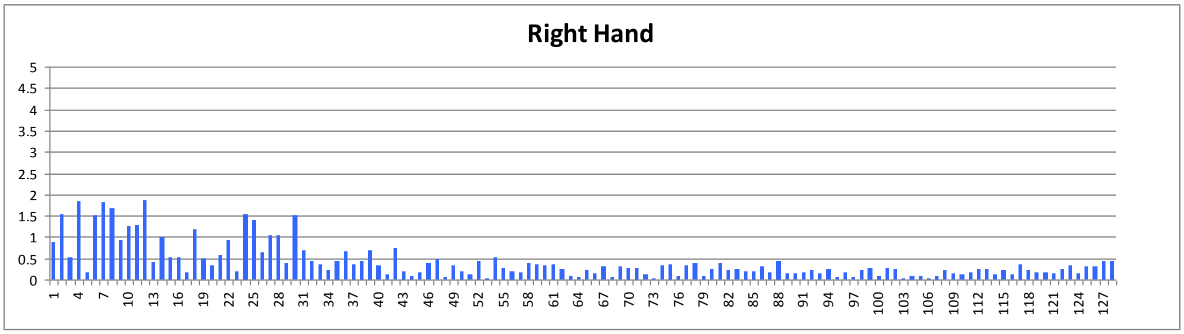
- Combined dimension: 1.8209

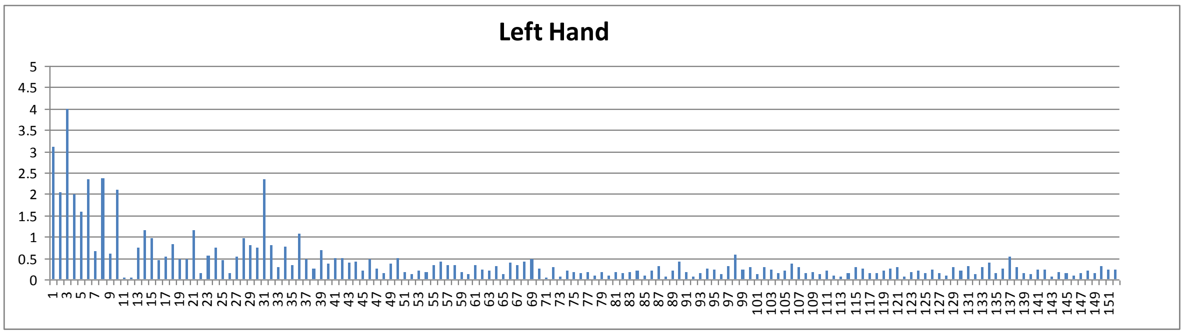


Attractor Plot

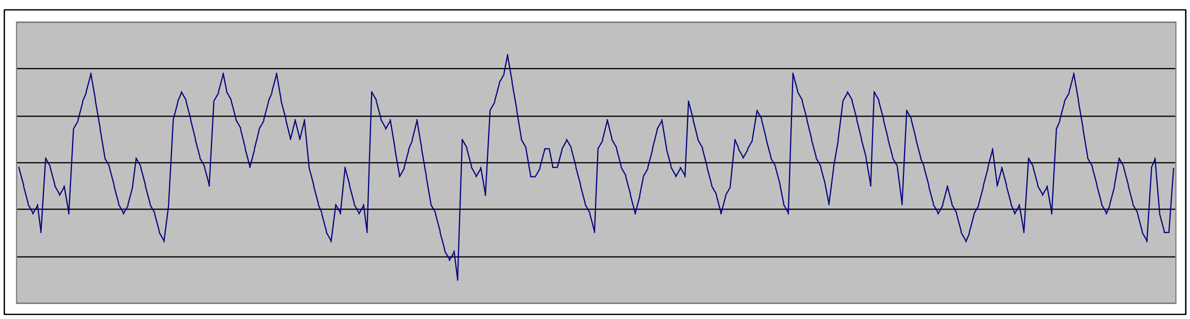


Spectral Analysis

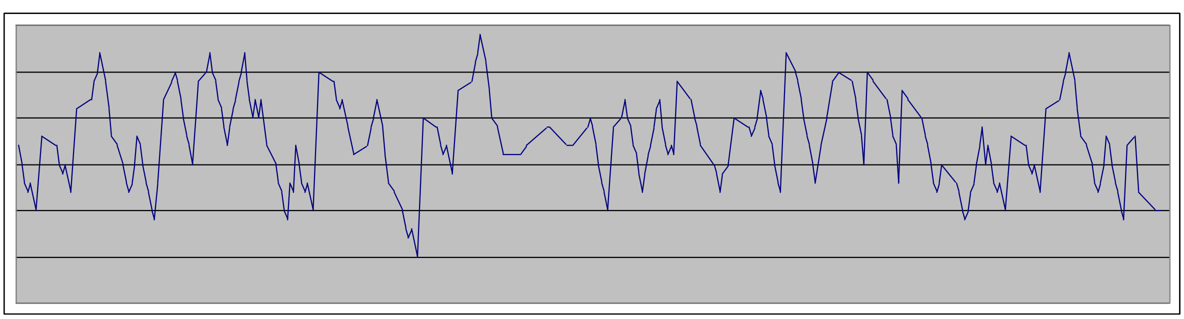




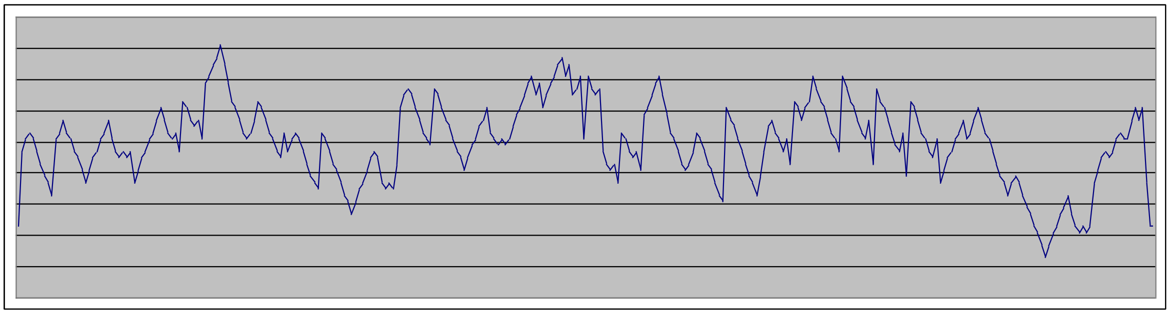
Inverse Transforms



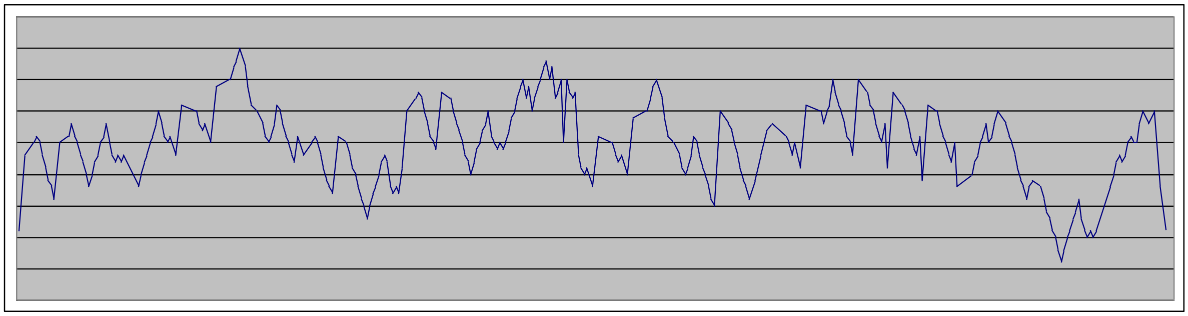
Inverse Transform (Right Hand)



Original Graph (Right Hand)



Inverse Transform (Left Hand)



Original Graph (Left Hand)

The two measurements for dimension are remarkably close.

The attractor graph shows an interesting orbital pattern in that most of the wider orbits are on the right side of the diagonal line, as only two of the larger orbits are on the left side of the diagonal. This is due to the fact that almost all of the larger intervals are descending. Most of the notes are spread evenly along the diagonal line, although there appears to be one weak basin of attraction in the right hand.

The harmonic spectrum of the right hand has an interesting pattern. While there are no unusually high amplitude harmonics, the amplitudes remain fairly high as far as the thirtieth harmonic, after which they do settle down, and there is no more significant activity.

The harmonic spectrum of the left hand has a similar pattern, although there are two high amplitudes at the first and third harmonics, and two extremely low amplitudes at the eleventh and twelfth harmonics. The amplitudes appear to settle down after this unusual behavior until a moderate spike in amplitude around the thirtieth harmonic, after which there is no more significant activity.

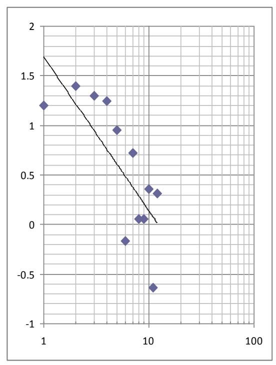
Two-Part Invention in G Major, No. 10, BWV 781 (1723)

Dimension, Madden

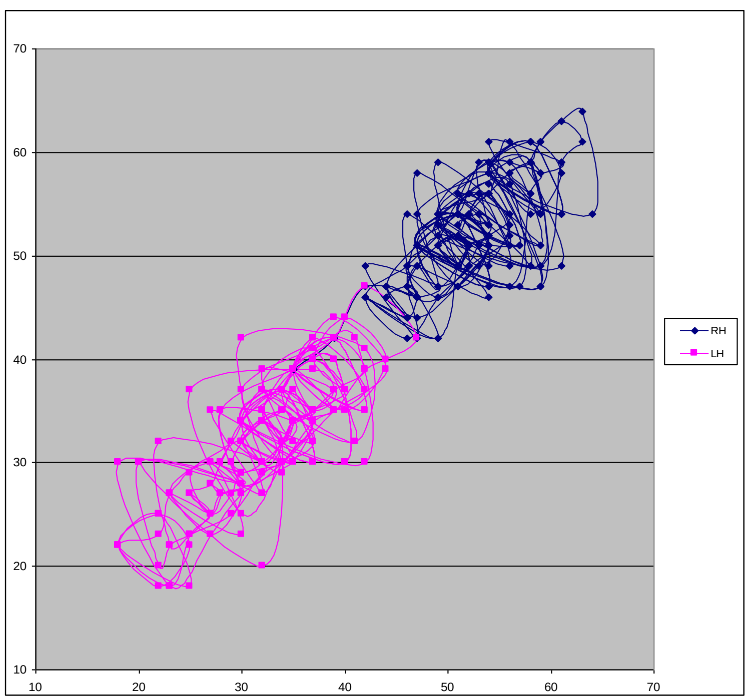
- Right hand: 1.2312
- Left hand: 1.2354

Dimension, Hsu

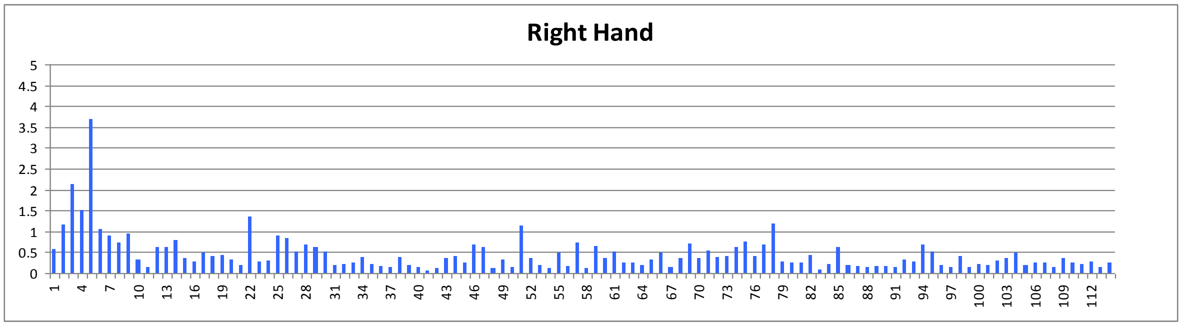
- Combined dimension: 1.5554

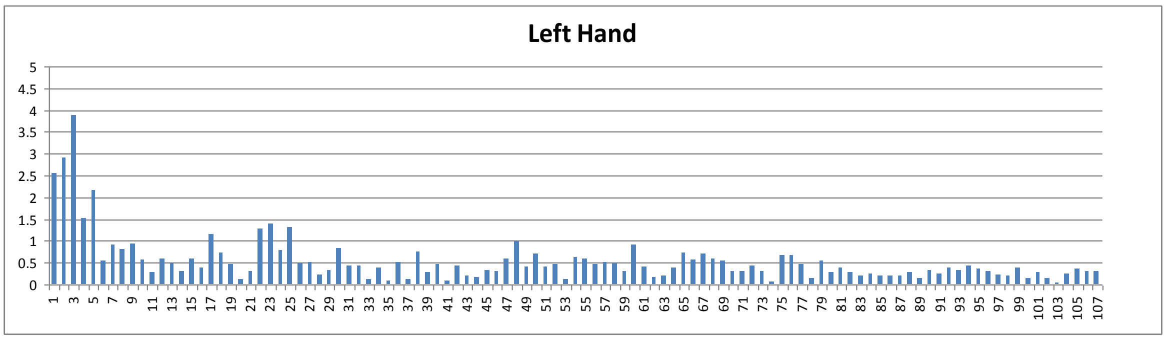


Attractor Plot

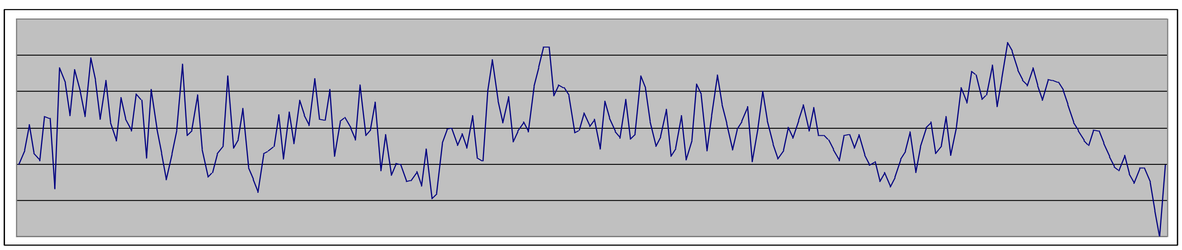


Spectral Analysis

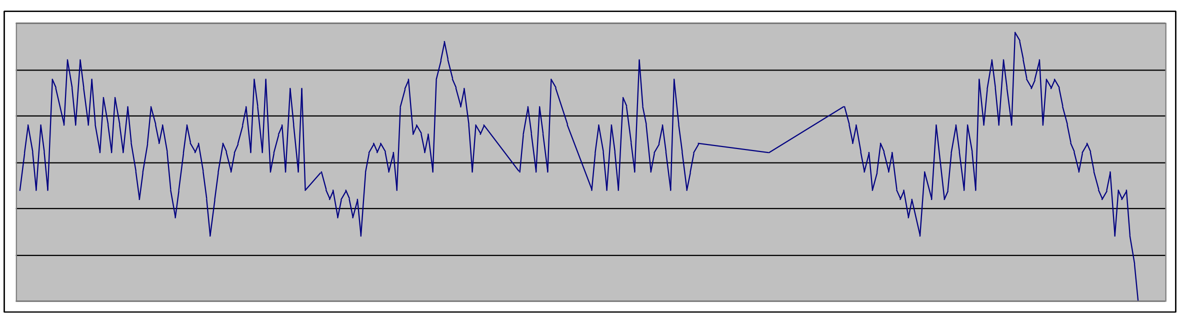




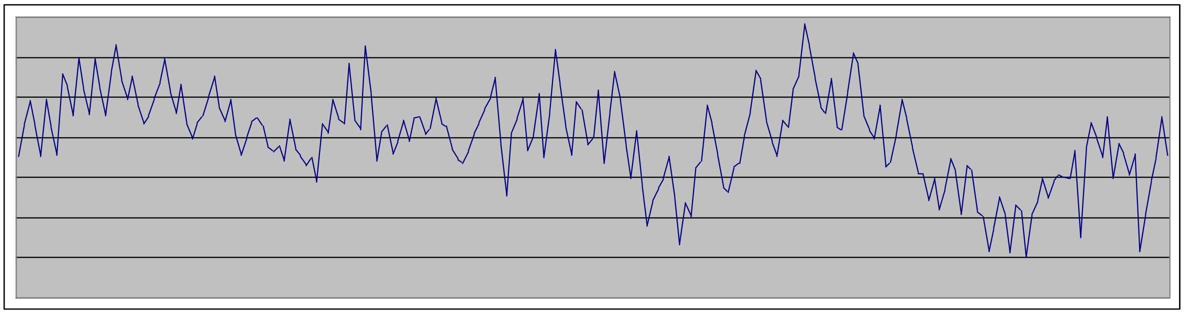
Inverse Transforms



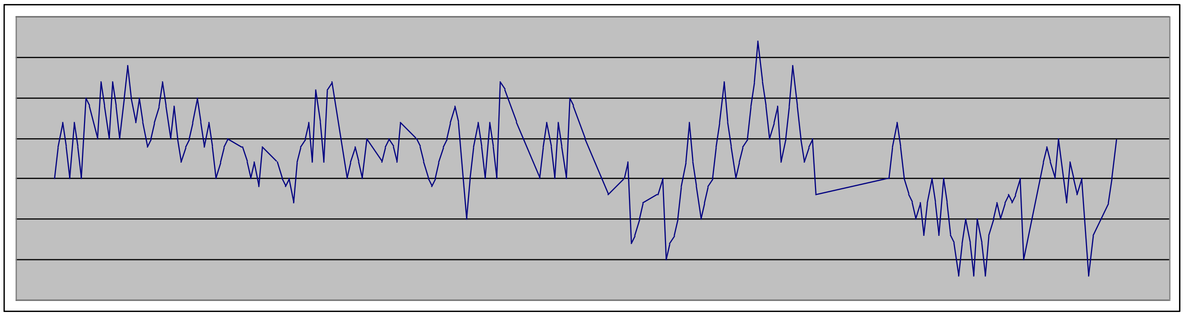
Inverse Transform (Right Hand)



Original Graph (Right Hand)



Inverse Transform (Left Hand)



Original Graph (Left Hand)

The attractor graph looks very similar for both hands with only a suggestion of a couple of weak attractors within the graph. There is an interesting break between the left and right hand lines in the attractor graph, indicating that the ranges of the voices overlap very little.

The harmonic spectrum of the right hand begins with a low amplitude, which is followed by a gradual overall growth in amplitude over the next few harmonics ending with a high amplitude at the fifth harmonic. After that the amplitudes settle down, although there is some amount of energy in the upper harmonics.

The harmonic spectrum of the left hand begins with a fairly high amplitude harmonic, which is followed by two harmonics of even higher amplitude. After the third harmonic, the amplitudes gradually drop off, although some energy remains through the upper harmonics.

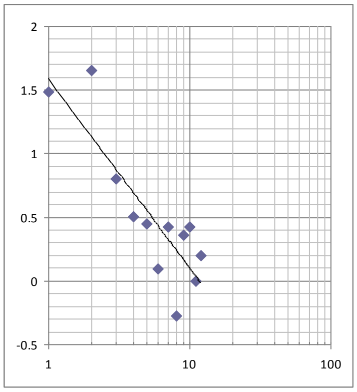
Two-Part Invention in G Minor, No. 11, BWV 782 (1723)

Dimension, Madden

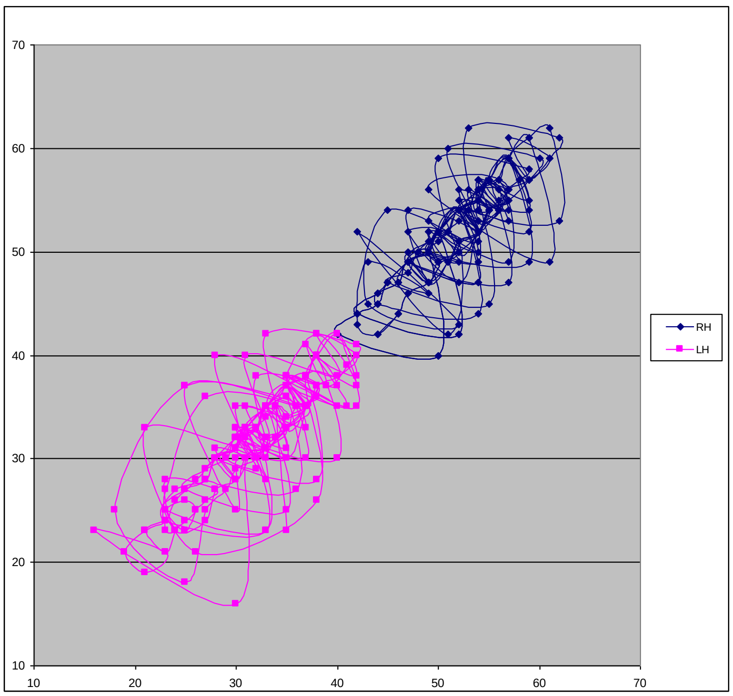
- Right hand: 1.1635
- Left hand: 1.1828

Dimension, Hsu

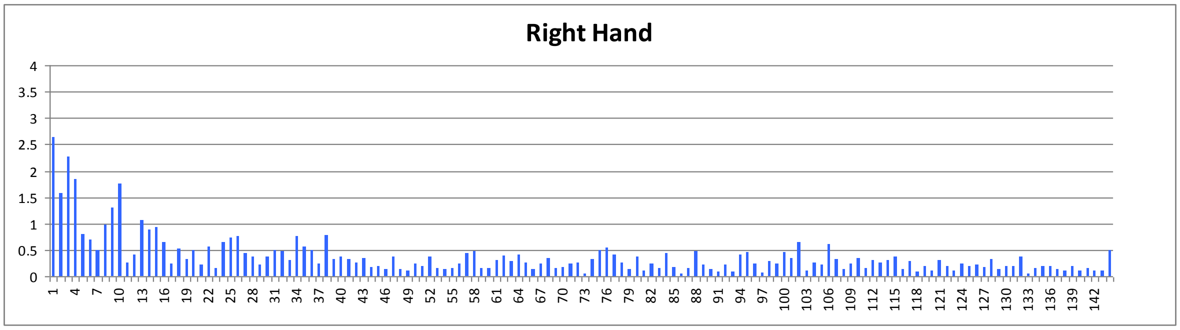
- Combined dimension: 1.4879

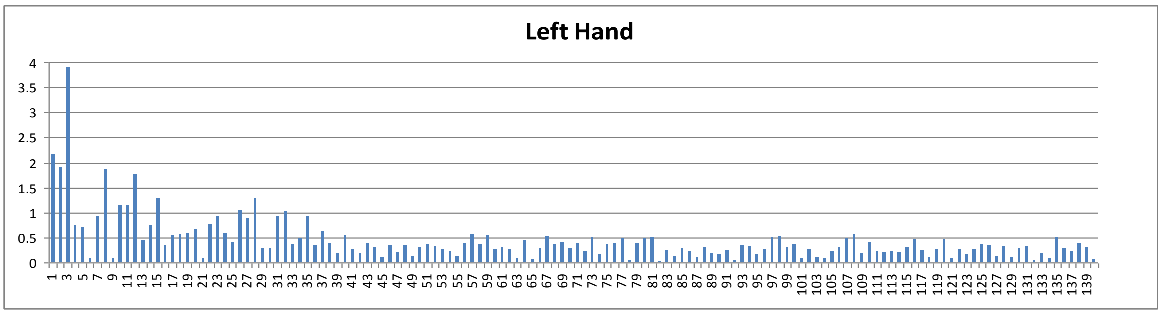


Attractor Plot

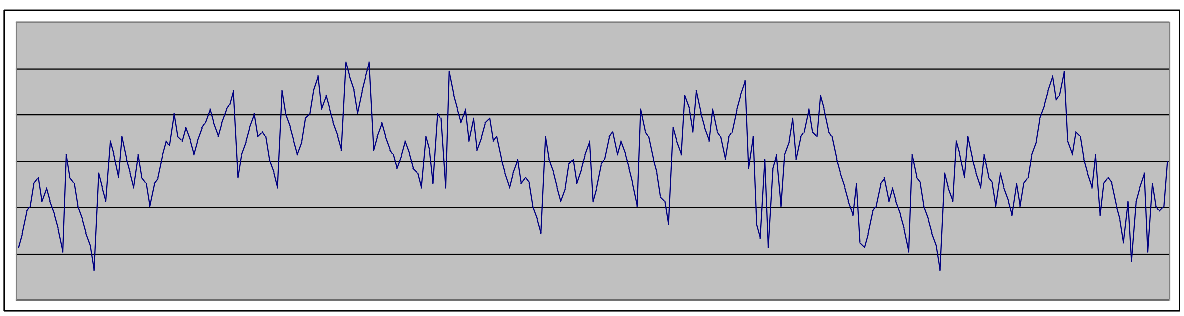


Spectral Analysis

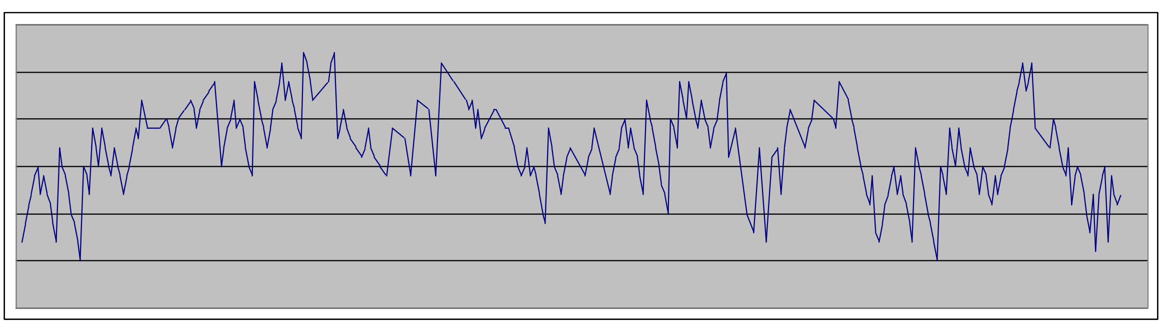




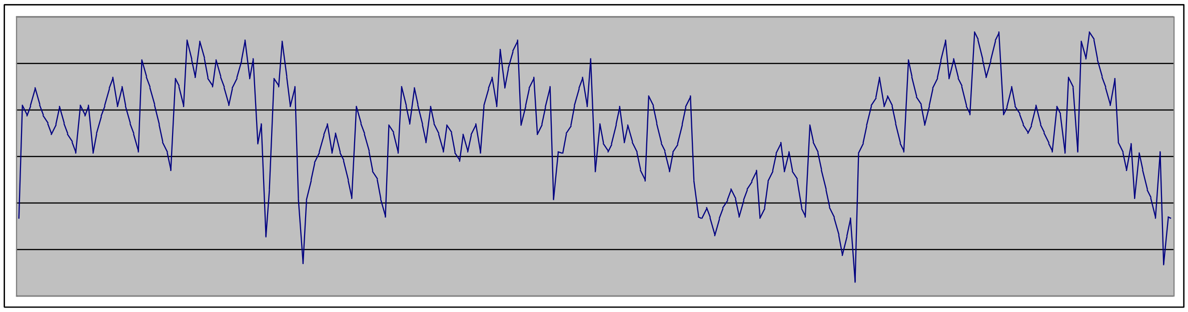
Inverse Transforms



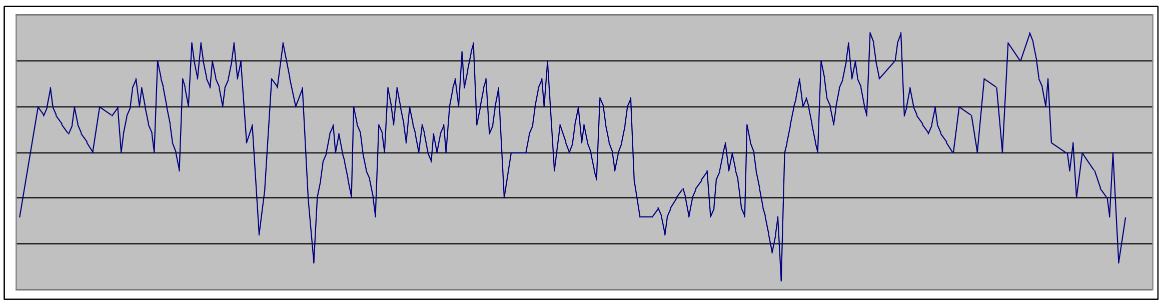
Inverse Transform (Right Hand)



Original Graph (Right Hand)



Inverse Transform (Left Hand)



Original Graph (Left Hand)

The slight difference in the dimensions is due to the occurrence of a few more large intervals in the left hand line.

The attractor graph shows an interesting break between the right hand and left hand lines, seeming to separate the two lines slightly in terms of keyboard geography. The wide orbits are due to the large intervals. There is one strong basin of attraction in each of the right hand and left hand lines.

The harmonic spectrum of the right hand is dominated by the first four harmonics, all of which have high amplitudes. After the fourth harmonic, the amplitudes generally decline until a suddenly high amplitude at the eleventh harmonic that is followed by a general decline, although there are a few more slight increases in amplitude. After each increase in amplitude, the amplitudes decline to a normal level again.

The harmonic spectrum for the left hand begins with two average-height harmonics, which are followed by a very high amplitude third harmonic. After the third harmonic there is an interesting pattern of randomly alternating high and low amplitude harmonics that gradually drop off to a normal level at about the thirty-fifth harmonic, after which there is no more significant activity.

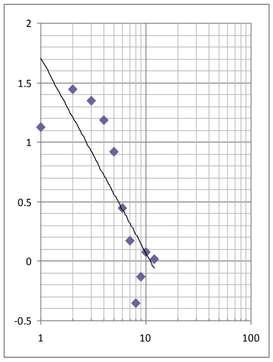
Two-Part Invention in A Major No. 12, BWV 783 (1723)

Dimension, Madden

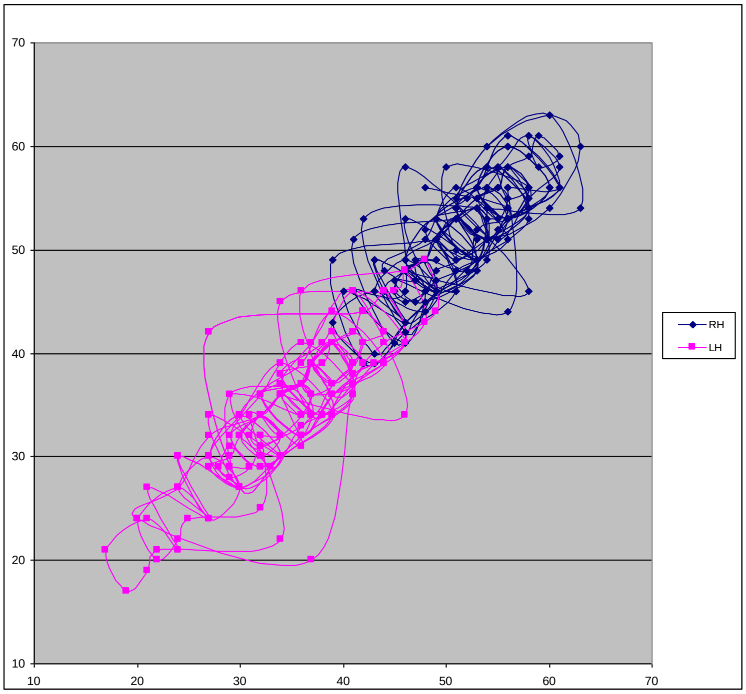
- Right hand: 1.1993
- Left hand: 1.1865

Dimension, Hsu

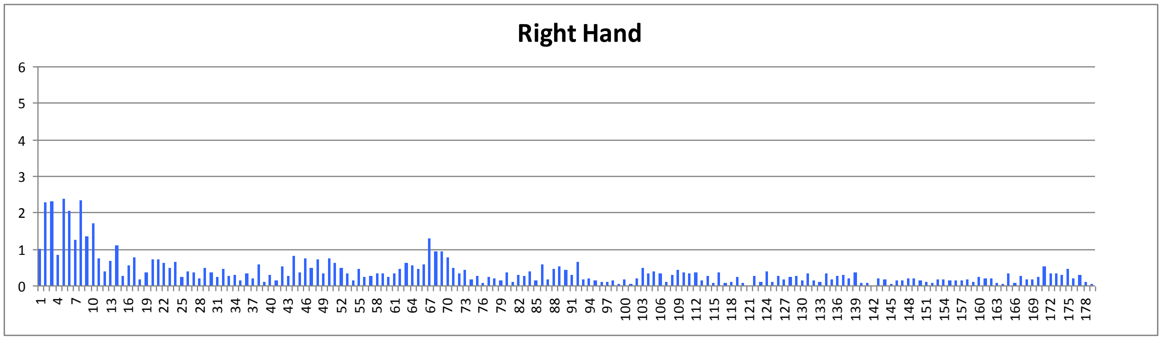
- Combined dimension: 1.6404

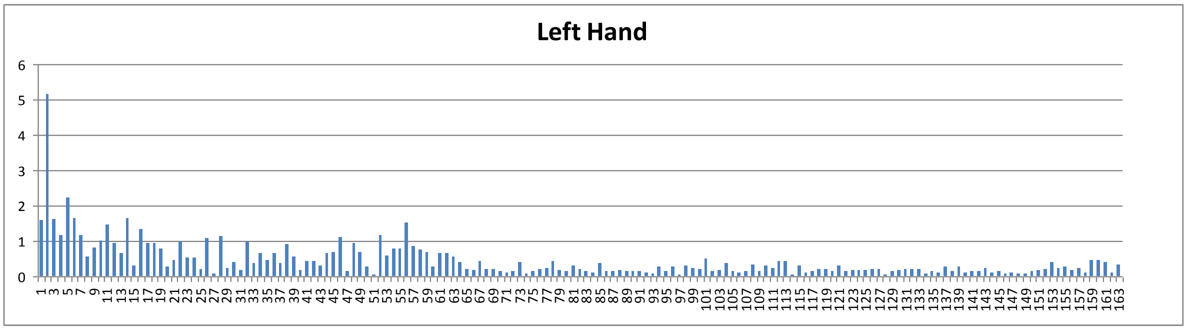


Attractor Plot

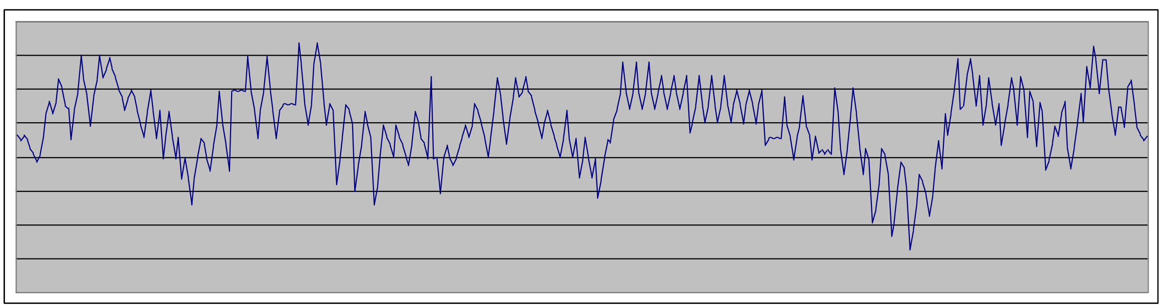


Spectral Analysis

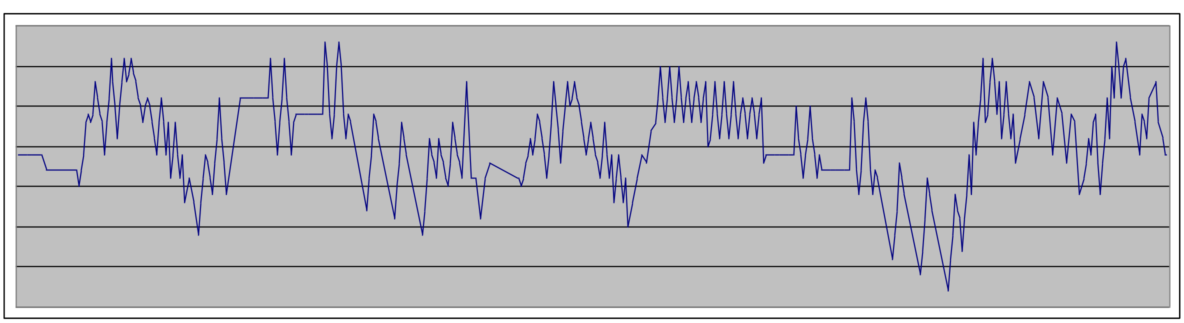




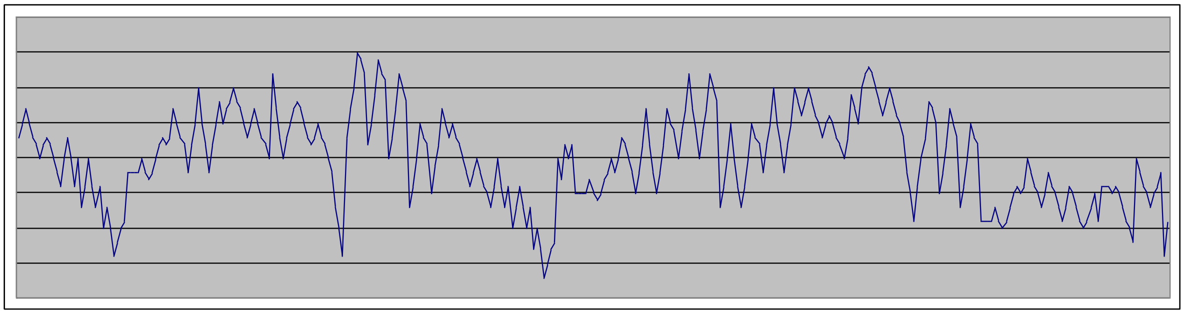
Inverse Transforms



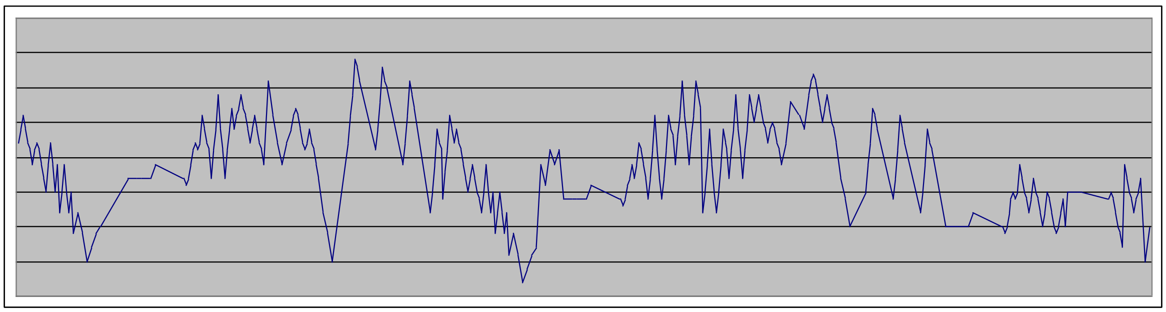
Inverse Transform (Right Hand)



Original Graph (Right Hand)



Inverse Transform (Left Hand)



Original Graph (Left Hand)

This attractor plot, in contrast to some of the other plots, shows a fairly wide interval of overlap between the left and right hands. The correlation along the diagonal line, while clear, is still somewhat weak, especially at the lower end of the left hand spectrum where the notes are not as tightly wound around the diagonal line. Although there are some indications of basins of attraction along the diagonal line, they are weak and unclear, since most of the notes are spread evenly throughout the graph.

The harmonic spectrum for the right hand has no significantly high amplitude harmonics. The first ten harmonics all have medium range amplitudes that vary slightly in height. After the tenth harmonic, the amplitudes seem to stabilize until a slight rise in amplitude around the sixty-seventh harmonic, after which the harmonics again drop off and stabilize with no activity in the upper harmonics.

The harmonic spectrum for the left hand has a high amplitude at the second harmonic, which is surrounded by harmonics with amplitudes in the medium range. Although there is a general (although slow) decrease in the amplitudes of the harmonics, the amplitudes do not really drop off and stabilize after about the fifty-fifth harmonic.

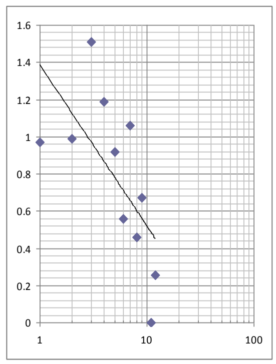
Two-Part Invention in A Minor, No. 13, BWV 784 (1723)

Dimension, Madden

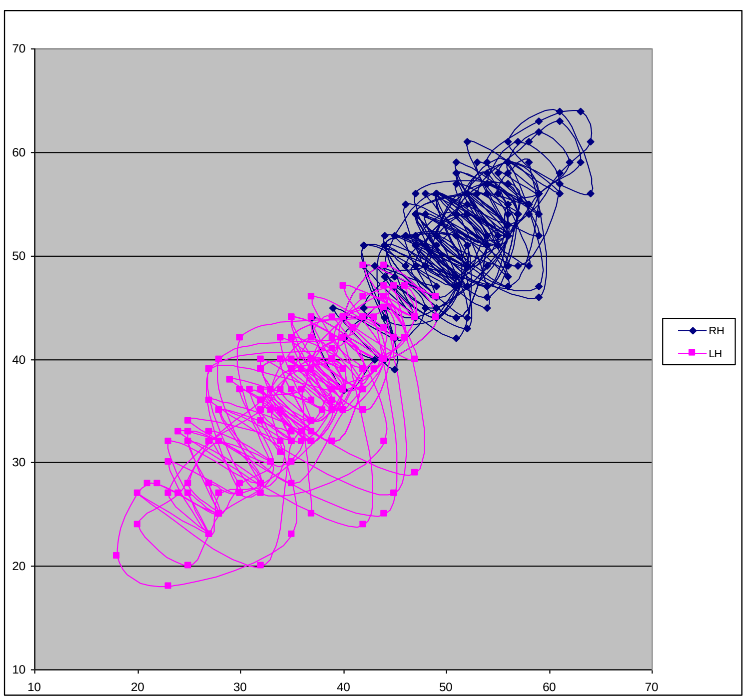
- Right hand: 1.2488
- Left hand: 1.2734

Dimension, Hsu

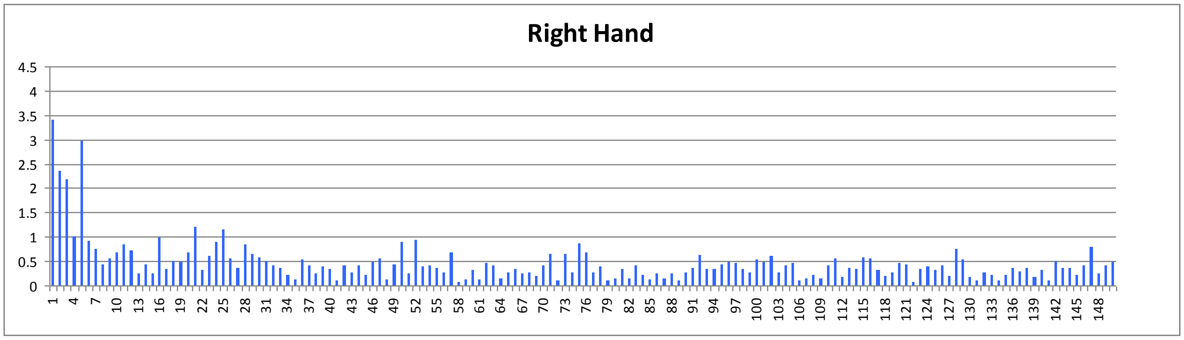
- Combined dimension: 0.8862

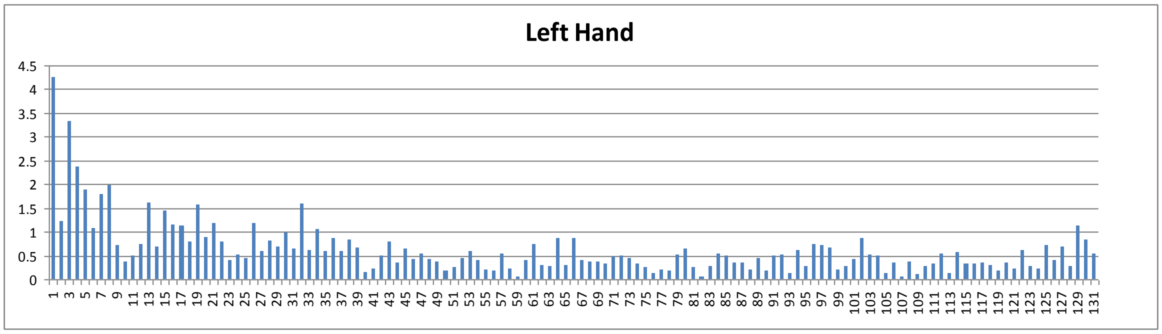


Attractor Plot

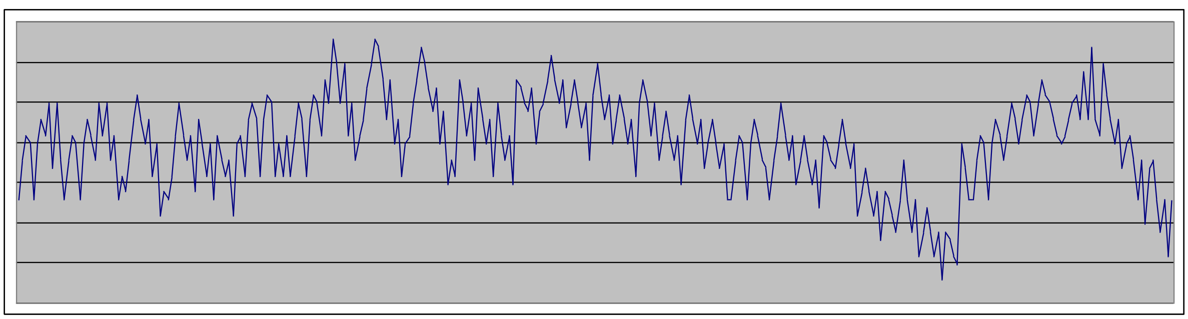


Spectral Analysis

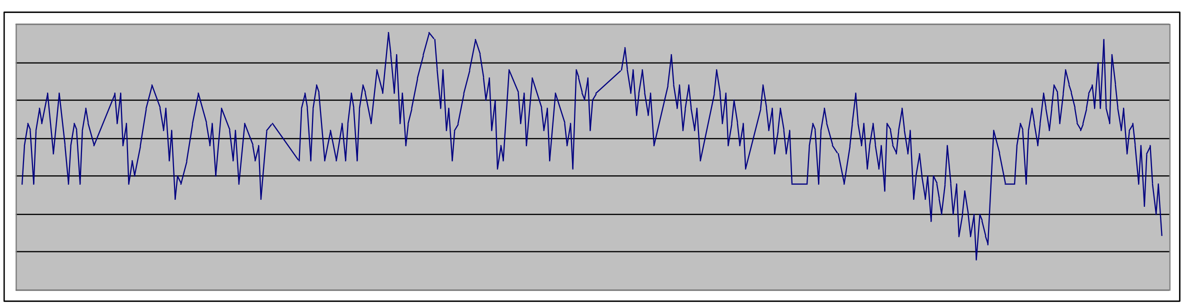




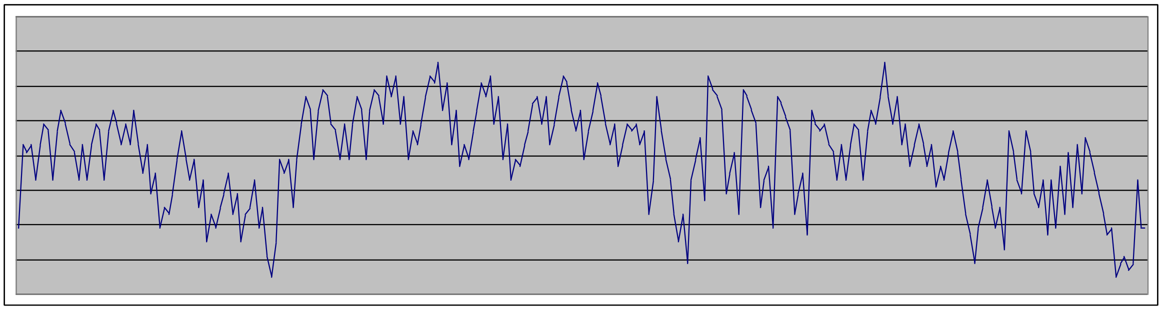
Inverse Transforms



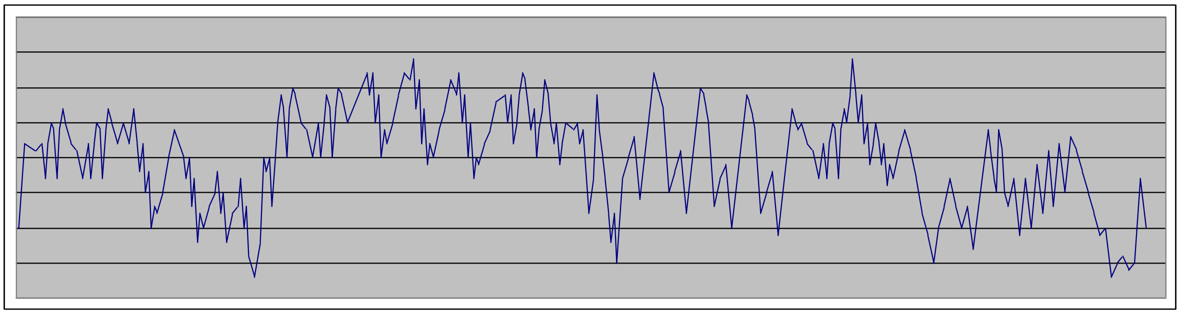
Inverse Transform (Right Hand)



Original Graph (Right Hand)



Inverse Transform (Left Hand)



Original Graph (Left Hand)

The Hsu dimension is abnormally low for this invention. I have been unable to determine the cause of this low dimension.

The attractor graph has an interesting orbital pattern due to the alternating note figuration used motivically throughout the piece, causing some of the orbits to rock back and forth linearly instead of circularly. The left hand has some wider orbits due to the larger intervals that occur in the left hand line.

The right hand harmonic spectrum begins with three high amplitude harmonics, after which there is a low amplitude harmonic. There follows a high amplitude spike at the fifth harmonic, after which the amplitudes gradually drop off. Although there are no significantly high amplitudes in the higher harmonics, there is more energy in the higher harmonics than has been typical, indicating a more complex melodic line.

The left hand harmonic spectrum begins with a high amplitude harmonic, which is followed by a low amplitude at the second harmonic, and another high amplitude at the third harmonic. After the third harmonic the amplitudes gradually drop off, although some energy remains throughout all of the upper harmonics.

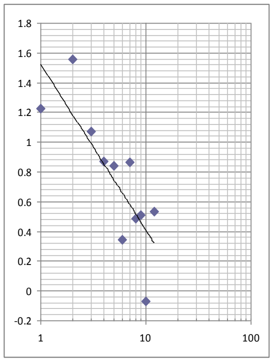
Two-Part Invention in Bb Major, No. 14, BWV 785 (1723)

Dimension, Madden

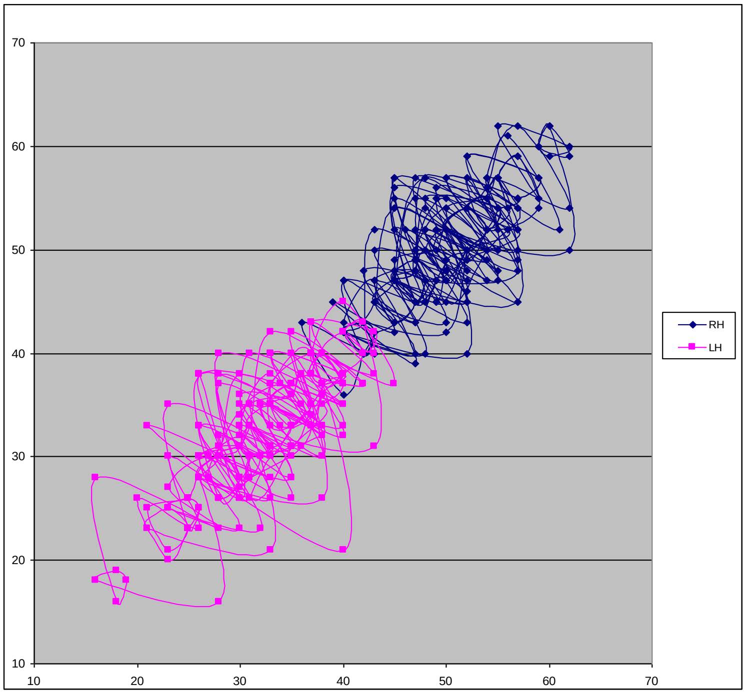
- Right hand: 1.2255
- Left hand: 1.2267

Dimension, Hsu

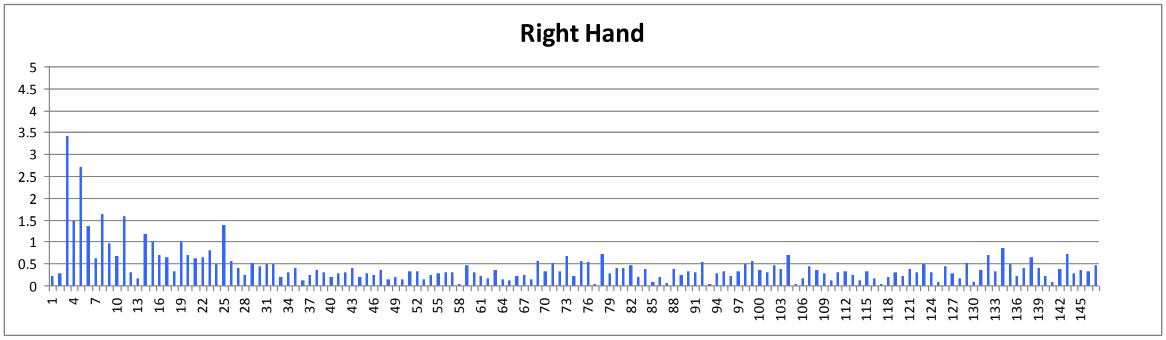
- Combined dimension: 1.1170

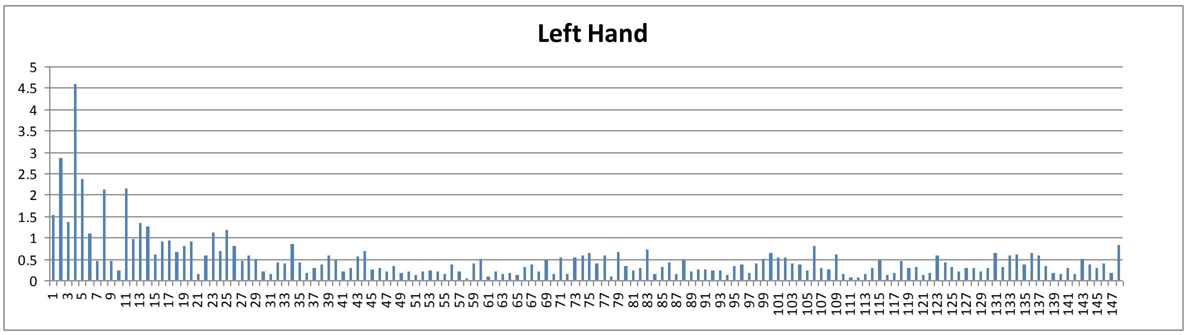


Attractor Plot

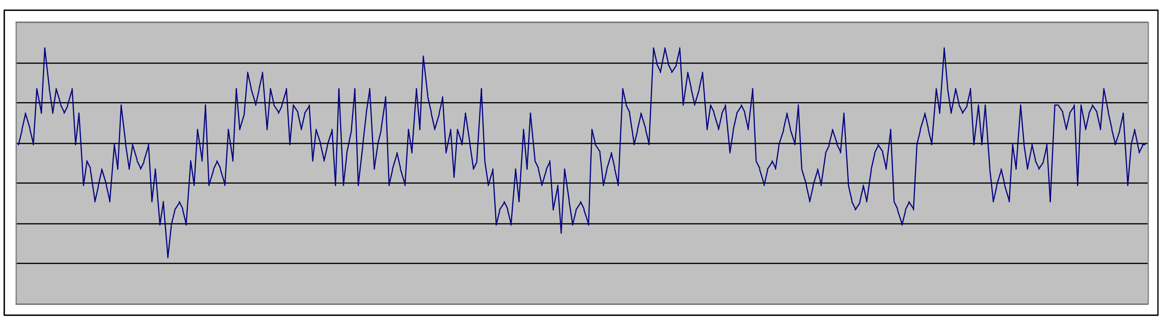


Spectral Analysis

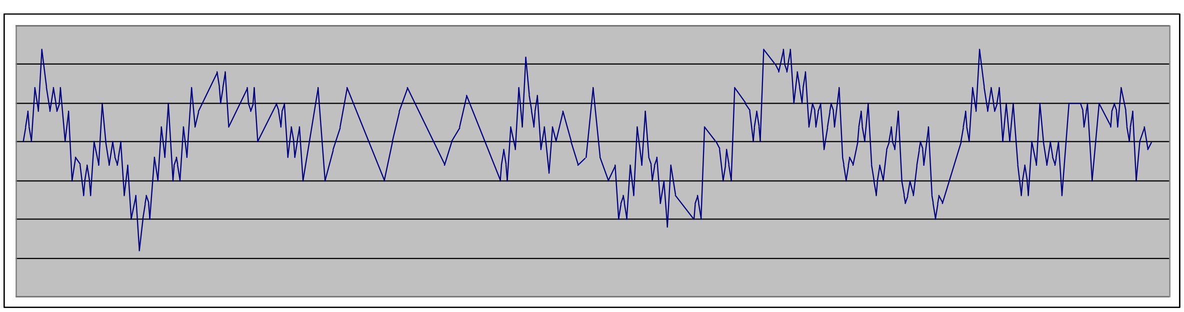




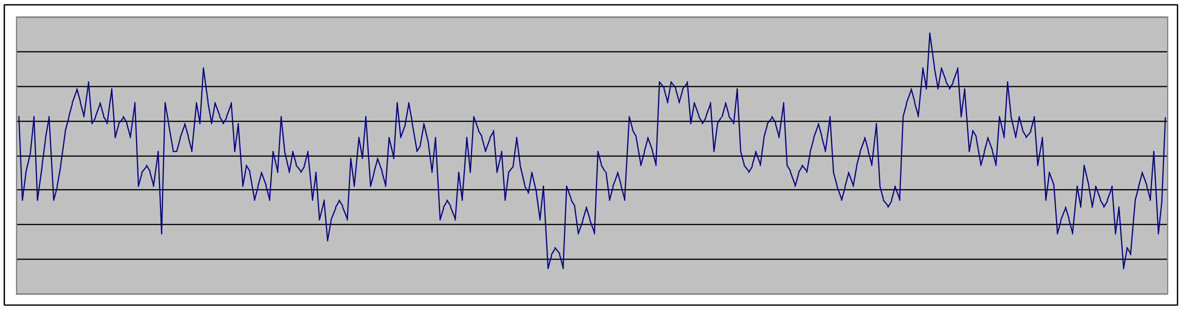
Inverse Transforms



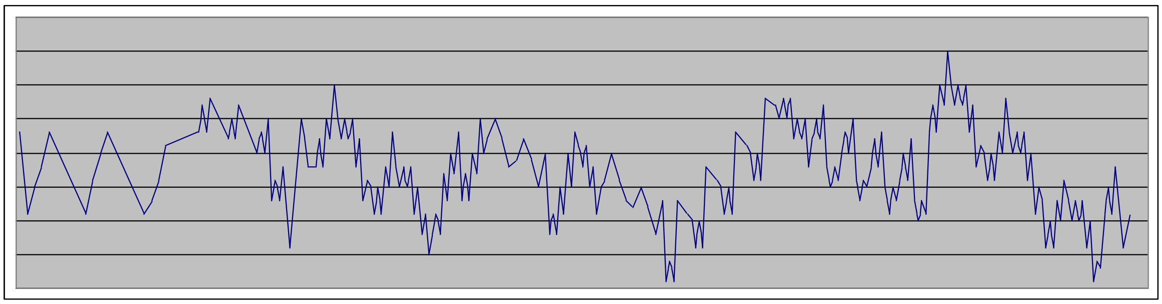
Inverse Transform (Right Hand)



Original Graph (Right Hand)



Inverse Transform (Left Hand)



Original Graph (Left Hand)

The dimensions of the two lines are almost equal. Although the left hand has more octave intervals than the right hand, the right hand has enough large intervals (m7 and M7) to make the dimensions almost equal.

The attractor graph has an interesting orbital shape with strongly defined basins of attraction.

The right hand harmonic spectrum begins with two unusually low amplitude harmonics, which is followed by a sudden jump in amplitude at the third harmonic. The amplitude drops at the fourth harmonic, and after rising again at the fifth harmonic, the amplitudes gradually drop off and stabilize, although there are several more unusually low amplitudes spread throughout the spectrum.

The left hand harmonic spectrum has medium-high amplitudes at the first and third harmonic, which alternate with high amplitudes at the second and fourth harmonics. The erratic alternation of high and low amplitudes continue (although the amplitudes are decreasing overall) through the thirtieth harmonic, after which the harmonics gradually drop and stabilize.

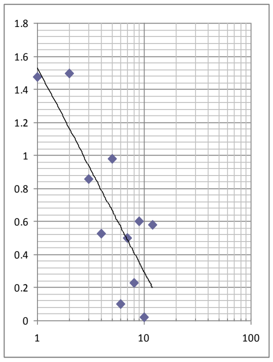
Two-Part Invention in B Minor, No. 15, BWV 786 (1723)

Dimension, Madden

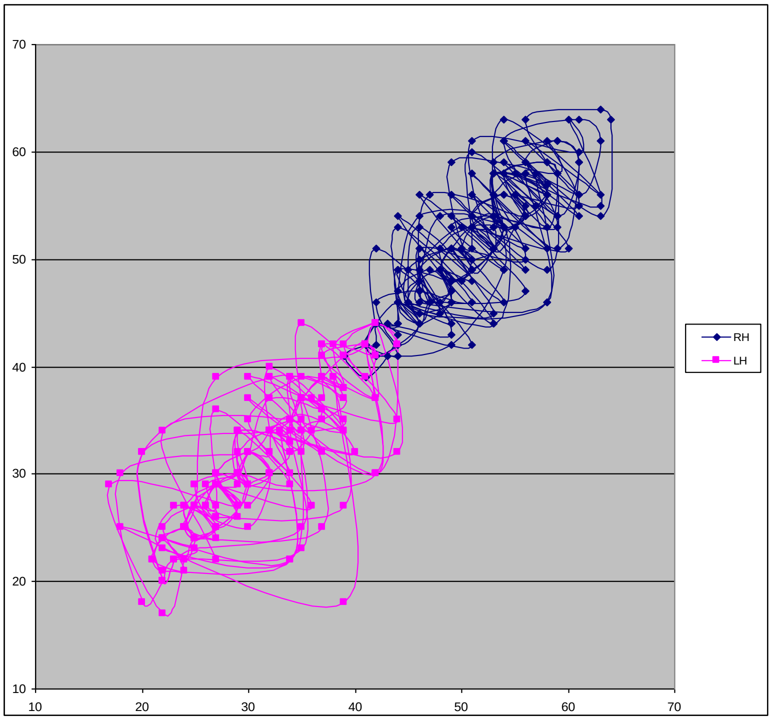
- Right hand: 1.2033
- Left hand: 1.2174

Dimension, Hsu

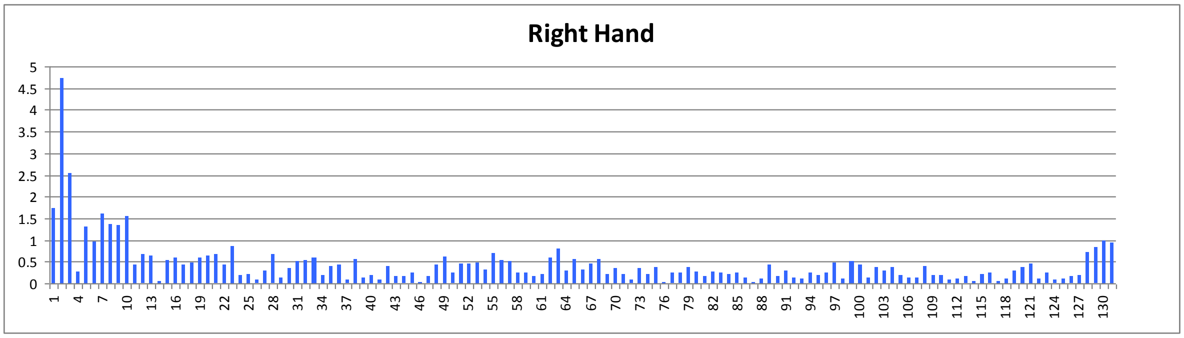
- Combined dimension: 1.2342

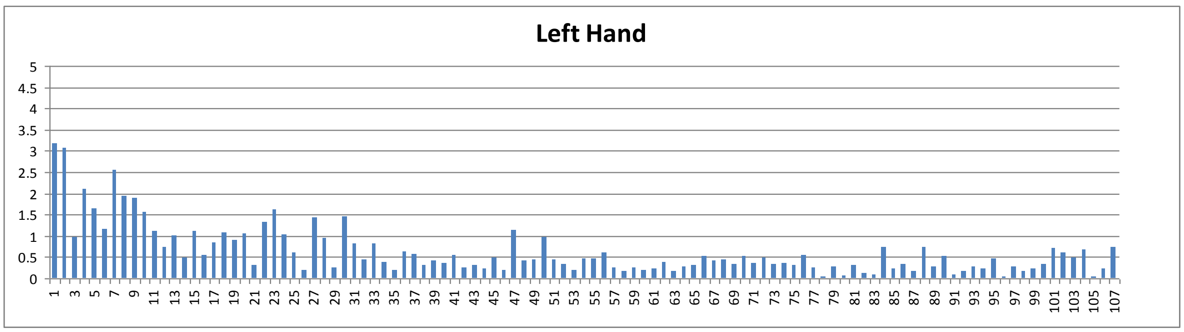


Attractor Plot

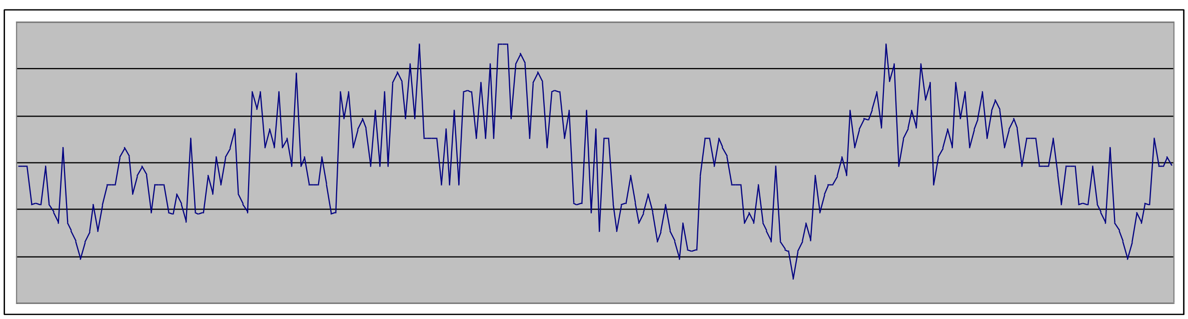


Spectral Analysis

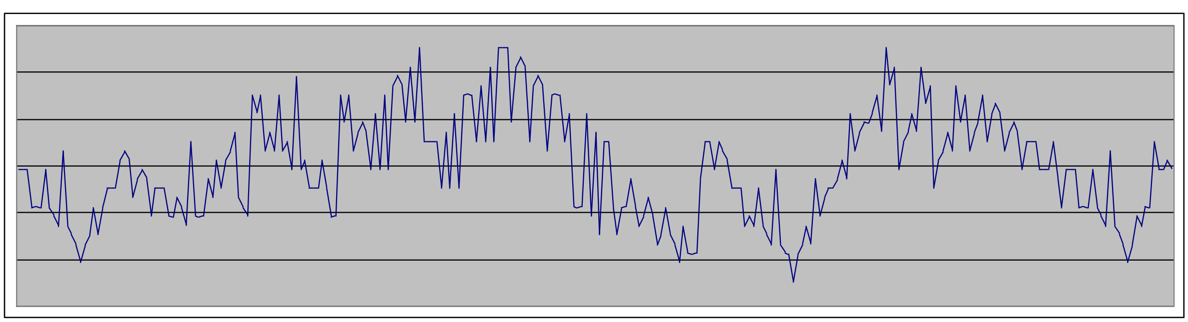




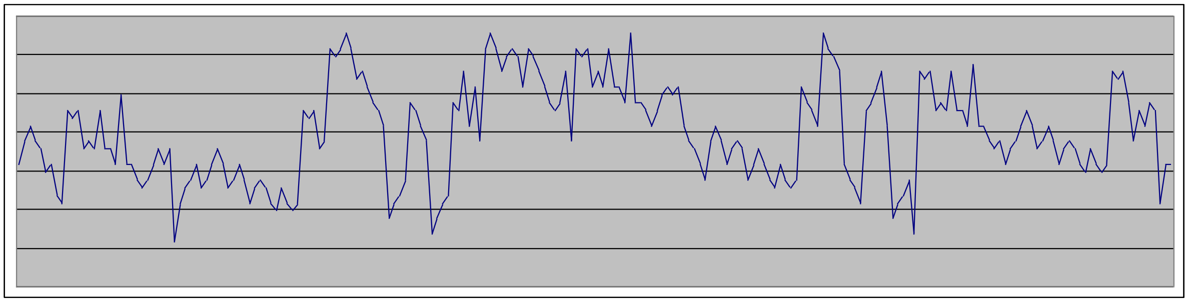
Inverse Transforms



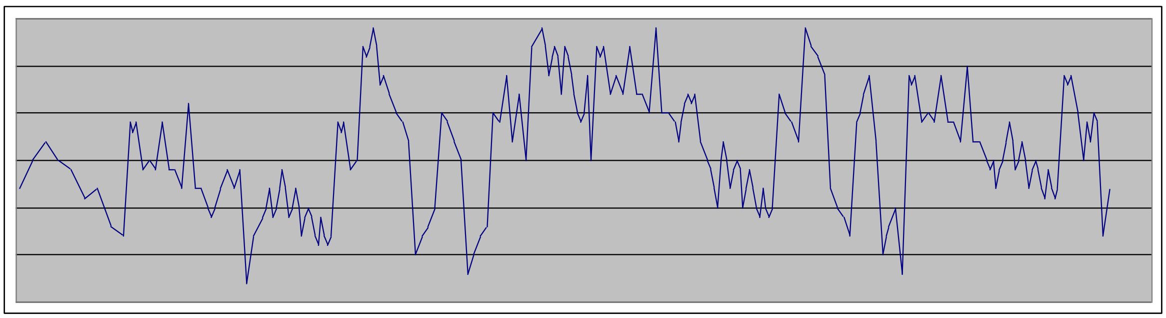
Inverse Transform (Right Hand)



Original Graph (Right Hand)



Inverse Transform (Left Hand)



Original Graph (Left Hand)

The attractor graph has three strong basins of attraction. While both lines show large intervals in their orbits, the left hand intervals are typically larger than the right hand, thus making the right hand spiral tighter around the center.

The harmonic spectrum of the right hand begins with a medium height amplitude at the first harmonic, which is followed by a high amplitude at the second harmonic. The amplitude drops at the third harmonic and drops to an extremely low amplitude at the fourth harmonic, after which the amplitudes rise again to a medium height, and then gradually drop off and stabilize, although there continue to be more low amplitudes throughout the upper harmonics. There is also an interesting, although slight, rise in the last four harmonics.

The harmonic spectrum of the left hand begins with two high amplitude harmonics that are followed by a drop at the third harmonic, a rise at the fourth and fifth harmonics, and then another drop at the sixth harmonic. After this alternation of high and low amplitudes, the amplitude rises yet again at the seventh harmonic, after which it gradually drops off and stabilizes. There is little energy in the upper harmonics except for a low spike at about the forty-seventh harmonic.

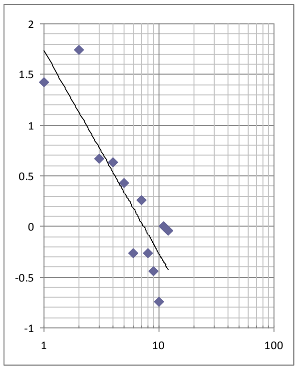
Three-Part Invention in C Major, No. 1, BWV 787 (1723)

Dimension, Madden

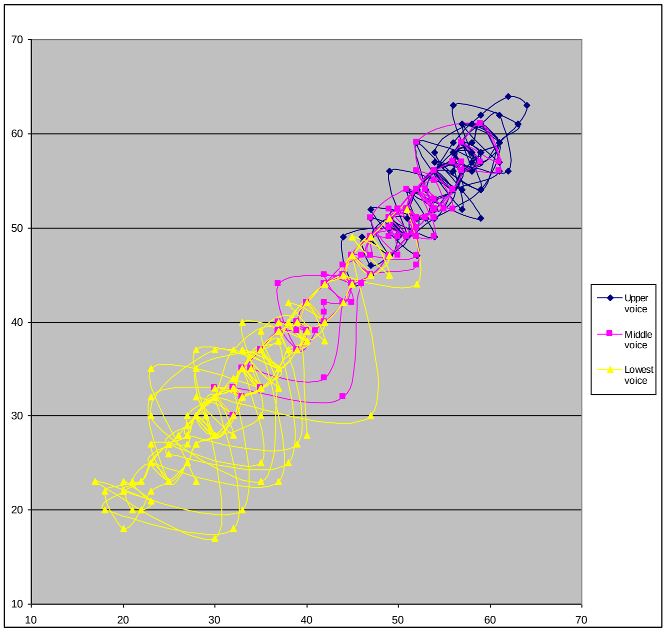
- Upper Voice: 1.1539
- Middle Voice: 1.1333
- Lower Voice: 1.1776

Dimension, Hsu

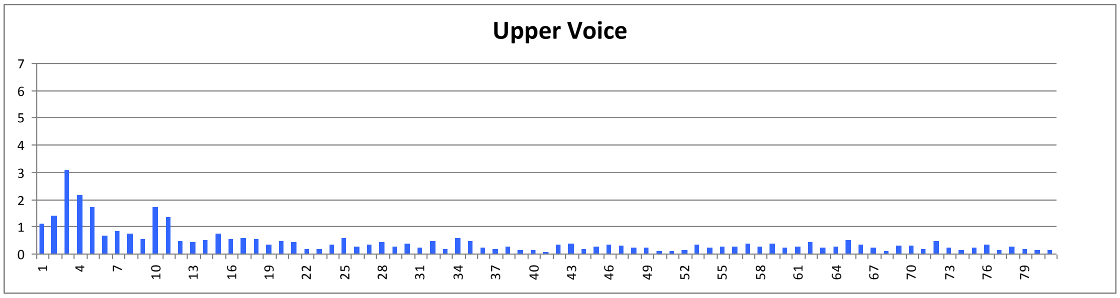
- Combined dimension: 2.0051

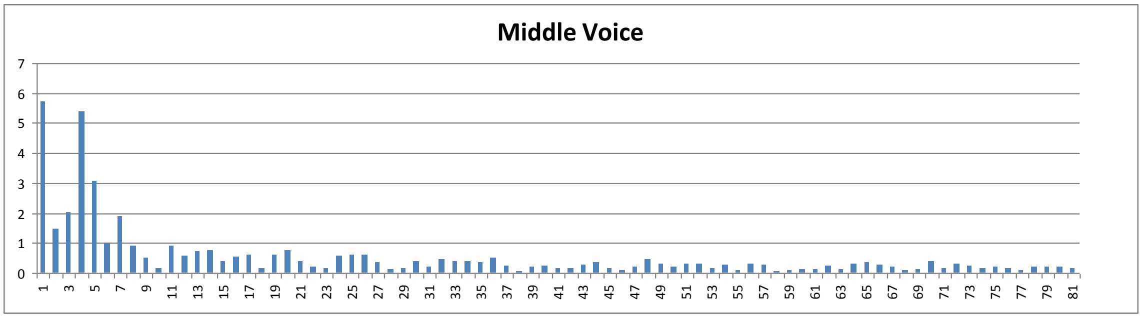


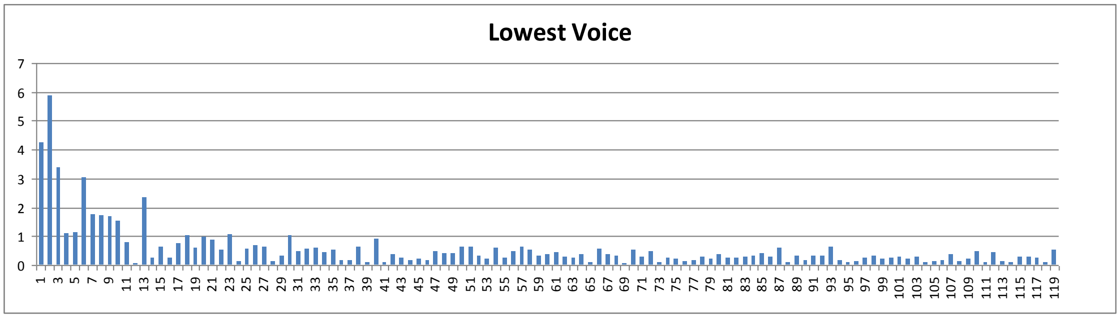
Attractor Plot



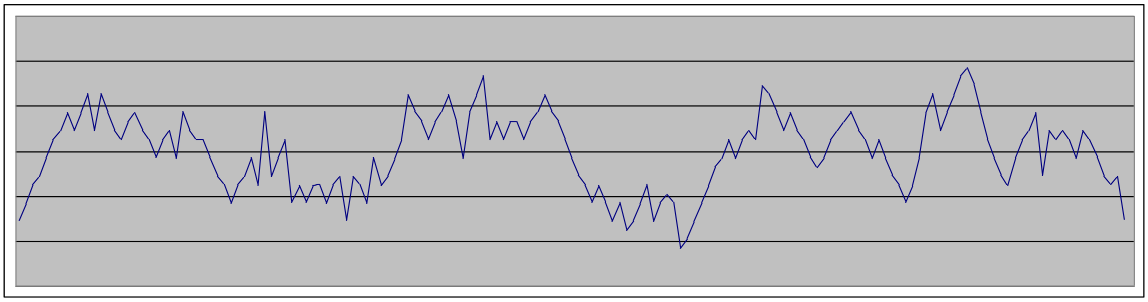
Spectral Analysis



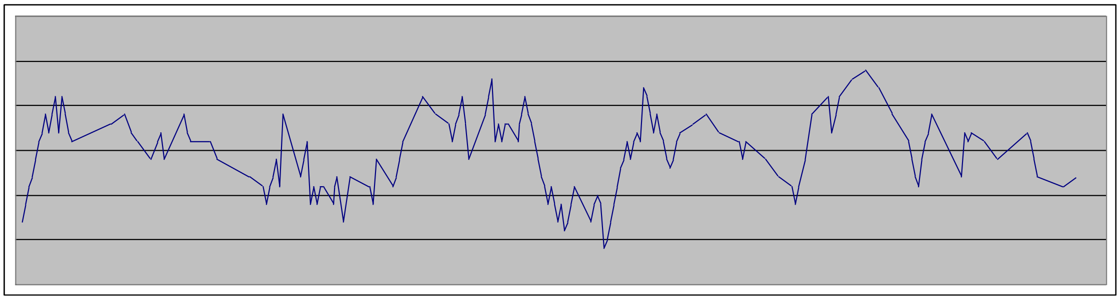




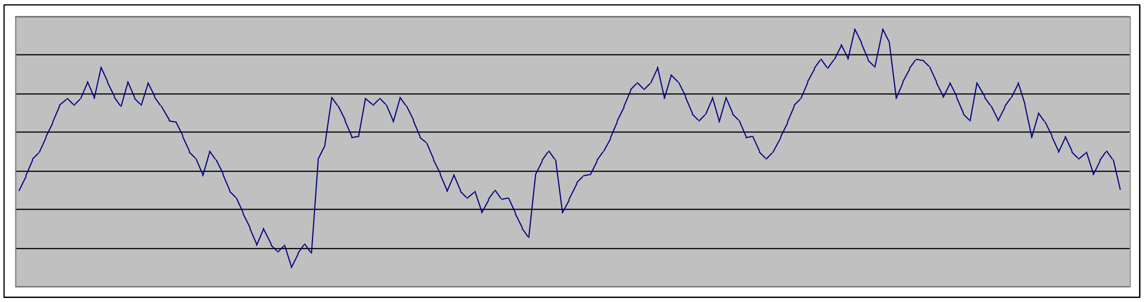
Inverse Transforms



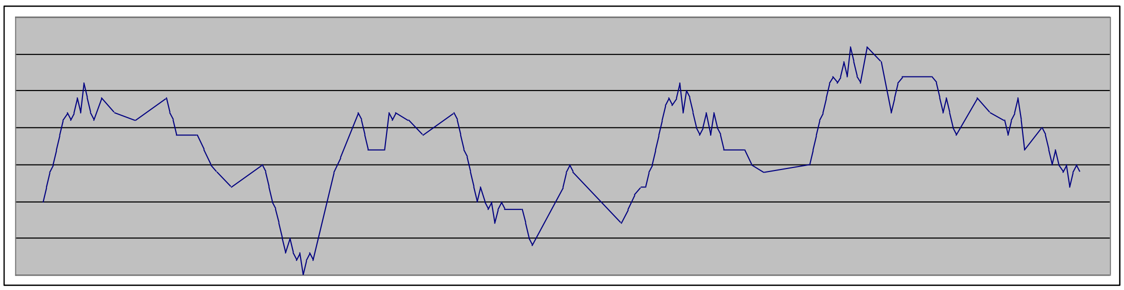
Inverse Transform (Upper Voice)



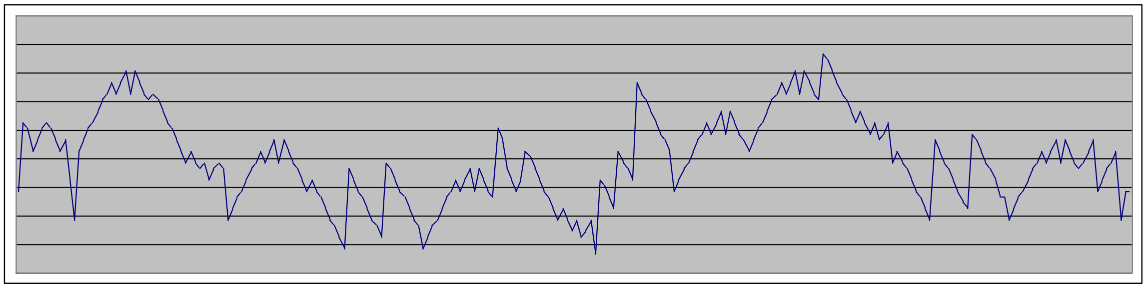
Original Graph (Upper Voice)



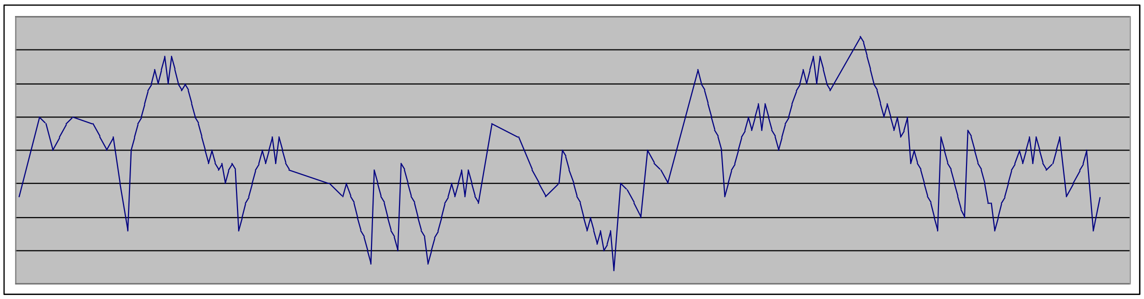
Inverse Transform (Middle Voice)



Original Graph (Middle Voice)



Inverse Transform (Lowest Voice)



Original Graph (Lowest Voice)

The Hsu dimension for this piece is abnormally high. While this dimension is still fractional, it falls far outside the expected range. This appears to be due to the scalar nature of the motive that is used as the basis for this piece.

The attractor graph has the typical correlation along the diagonal line, although the orbits are looser in the lowest voice because of the larger intervals that occur in the lowest voice. There are a few weak basins of attraction, but there are no strong basins of attraction within this graph.

The harmonic spectrum of the lowest voice has a slightly unusual distribution in that the amplitudes do not begin to stabilize until about the thirteenth harmonic, immediately after a very small amplitude at the twelfth harmonic.

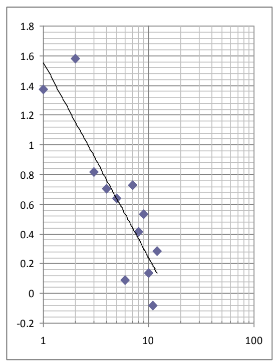
Three-Part Invention in C Minor, No. 2, BWV 788 (1723)

Dimension, Madden

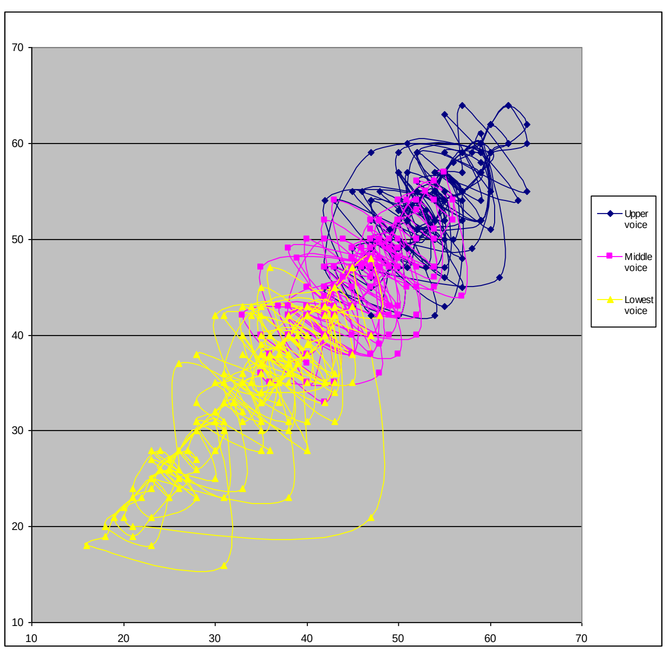
- Upper Voice: 1.1956
- Middle Voice: 1.2121
- Lower Voice: 1.2145

Dimension, Hsu

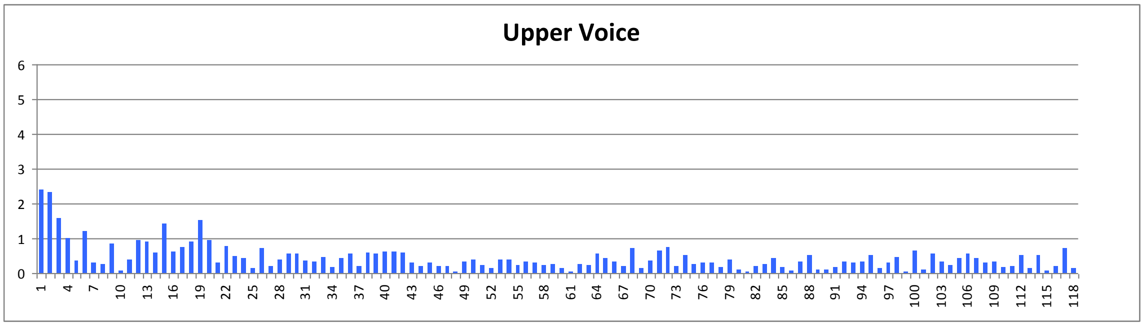
- Combined dimension: 1.3159

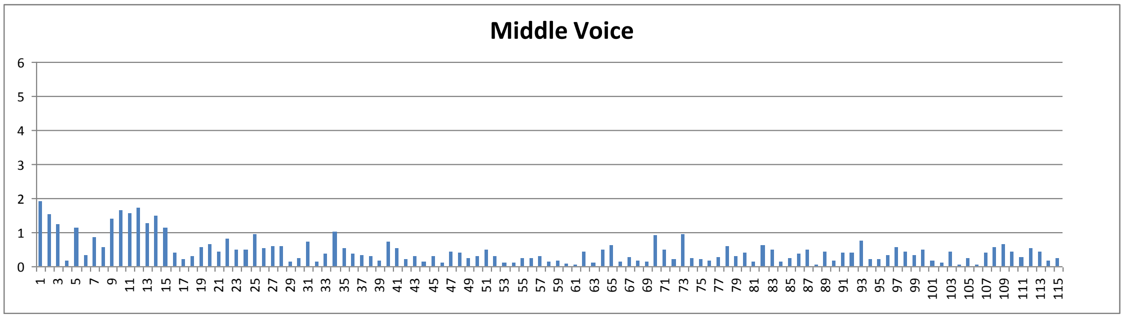


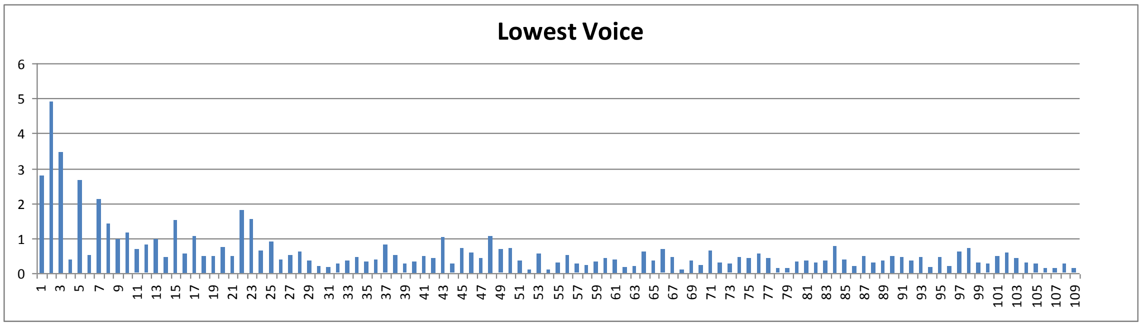
Attractor Plot



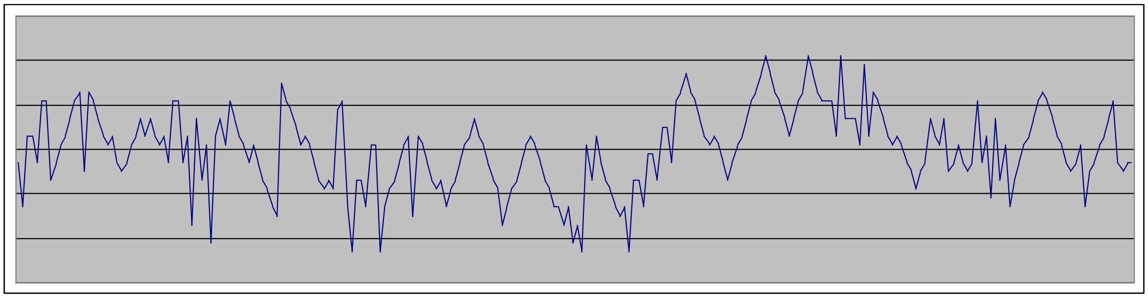
Spectral Analysis



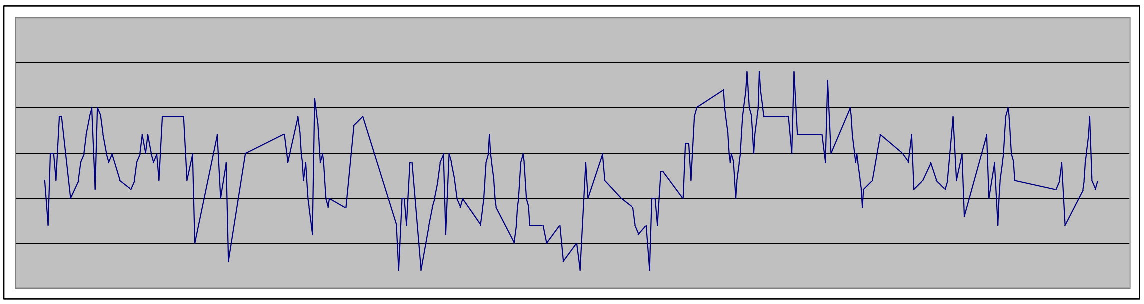




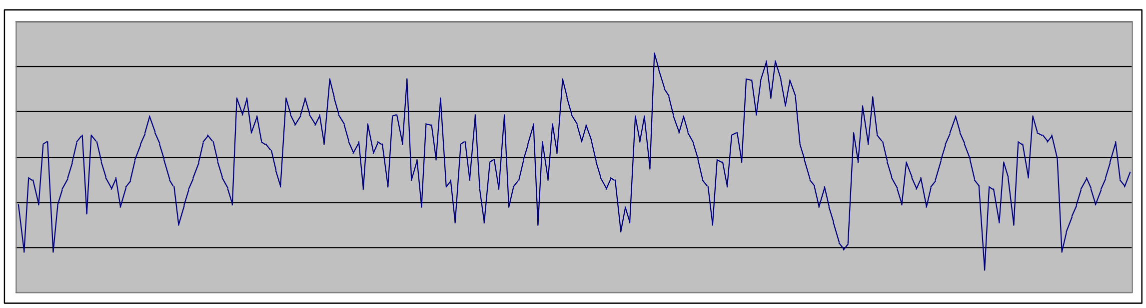
Inverse Transforms



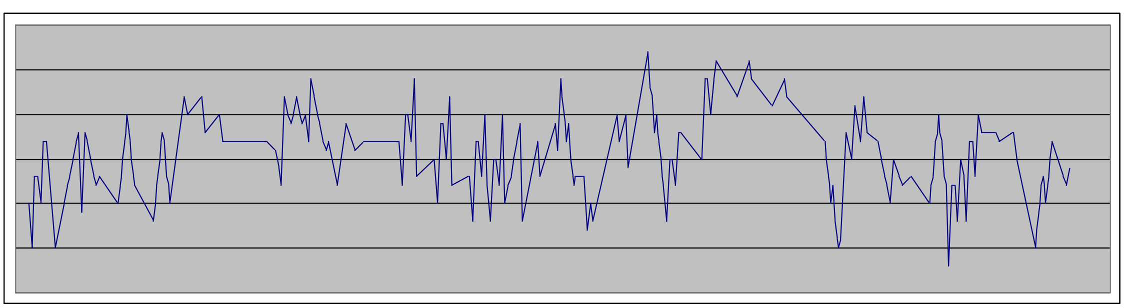
Inverse Transform (Upper Voice)



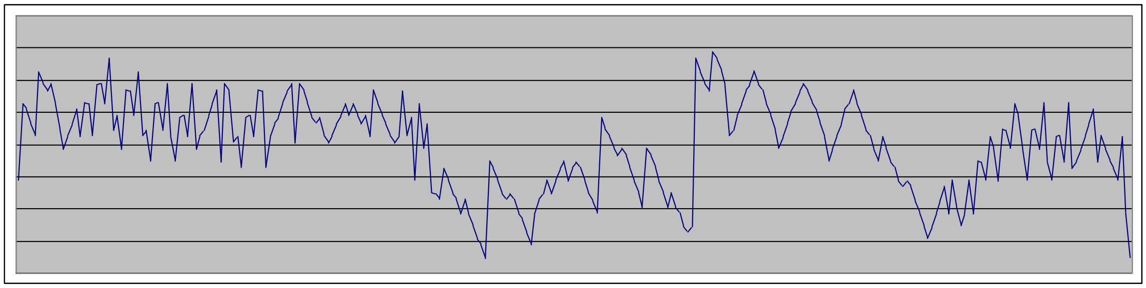
Original Graph (Upper Voice)



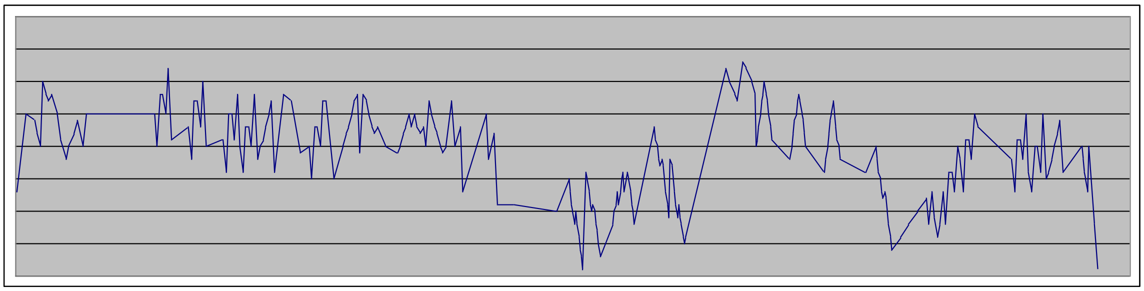
Inverse Transform (Middle Voice)



Original Graph (Middle Voice)



Inverse Transform (Lowest Voice)



Original Graph (Lowest Voice)

The attractor graph shows the typical correlation along the diagonal line. The few wide orbits in the lowest voice are due to some unusually large intervals in the lowest voice. There are four strong basins of attraction within this graph: one in the lowest voice, one in the middle voice, and two in the upper voice.

The harmonic spectrum of the lowest voice begins with three high amplitudes that are followed by three alternating low and high amplitudes, after which the amplitudes gradually drop off and stabilize.

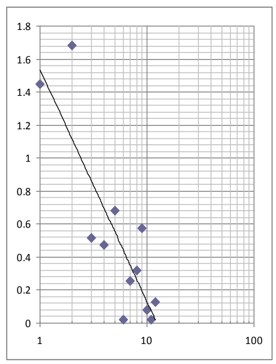
Three-Part Invention in D Major, No. 3, BWV 789 (1723)

Dimension, Madden

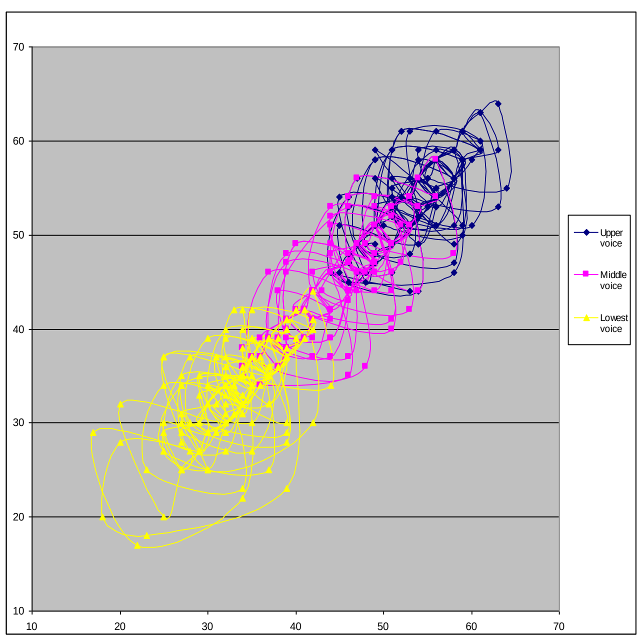
- Upper Voice: 1.1782
- Middle Voice: 1.1977
- Lower Voice: 1.1994

Dimension, Hsu

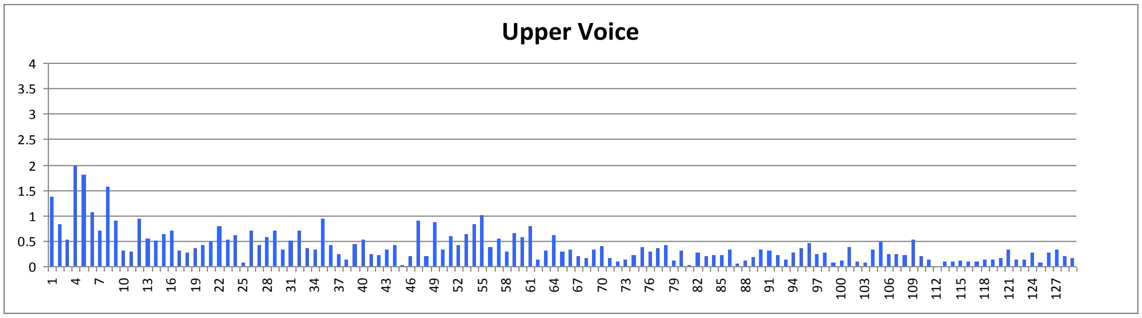
- Combined dimension: 1.4041

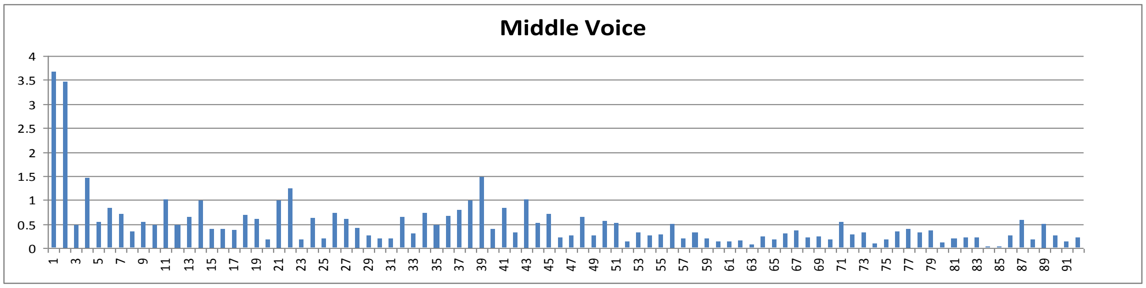


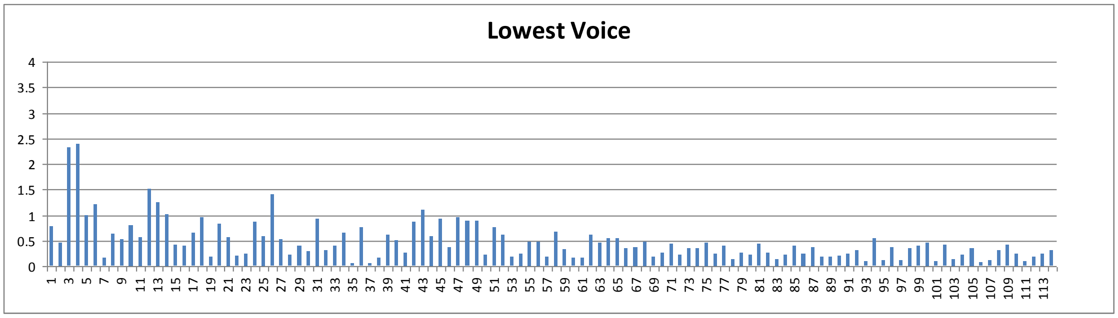
Attractor Plot



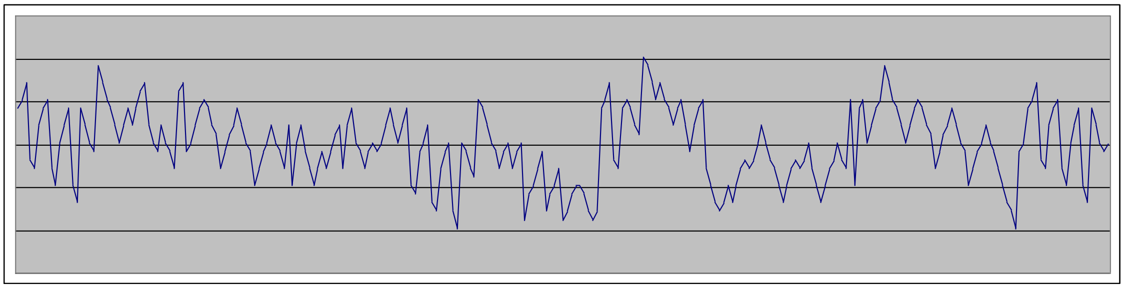
Spectral Analysis



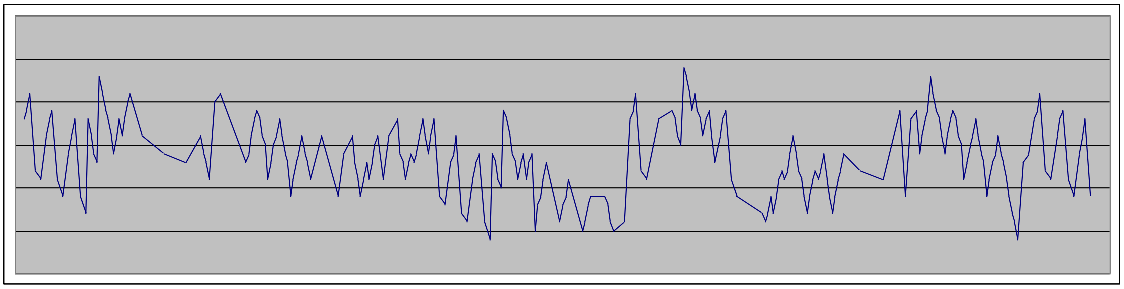




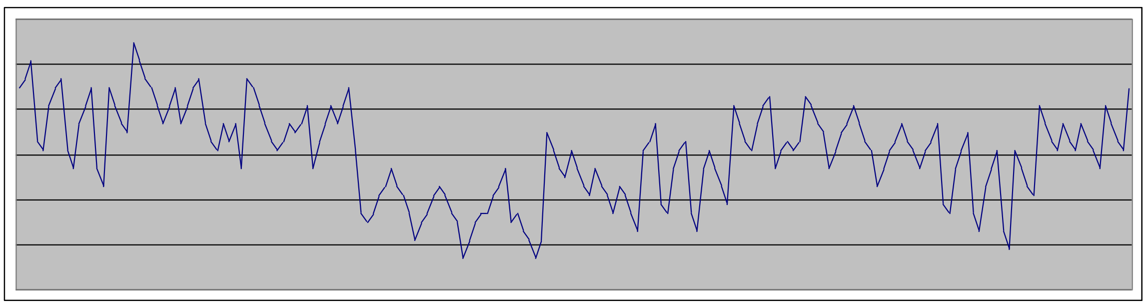
Inverse Transforms



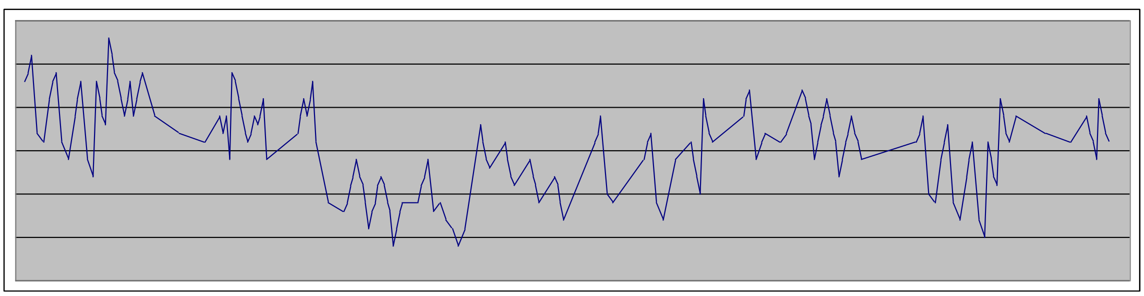
Inverse Transform (Upper Voice)



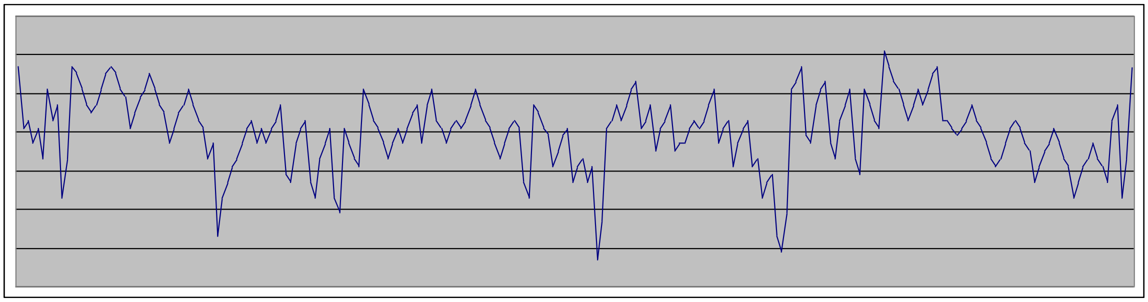
Original Graph (Upper Voice)



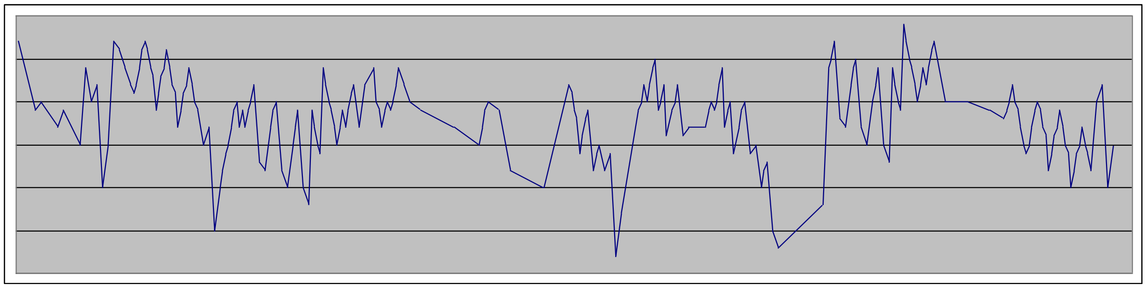
Inverse Transform (Middle Voice)



Original Graph (Middle Voice)



Inverse Transform (Lowest Voice)



Original Graph (Lowest Voice)

The attractor graph has a fairly even distribution throughout all the voices, with the large intervals that create the wide orbits evenly spread throughout all the voices. There are four moderately strong basins of attraction in this graph: one in the lowest voice, one in the middle voice, one in both the middle voice and the upper voice, and one in the upper voice.

The harmonic spectrums for all three voices have a similar distribution. The amplitudes in all three graphs do not stabilize and drop off until about the fiftieth harmonic. This is relatively late for the energy to finally drop. The middle voice spectrum has two high amplitude harmonics, but the upper and lowest voice spectrums do not have any outstandingly high amplitude harmonics.

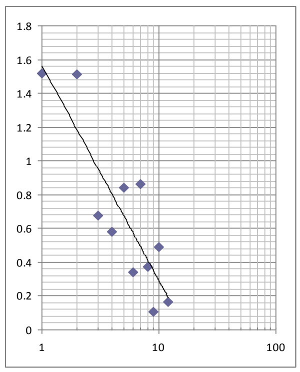
Three-Part Invention in D Minor, No. 4, BWV 790 (1723)

Dimension, Madden

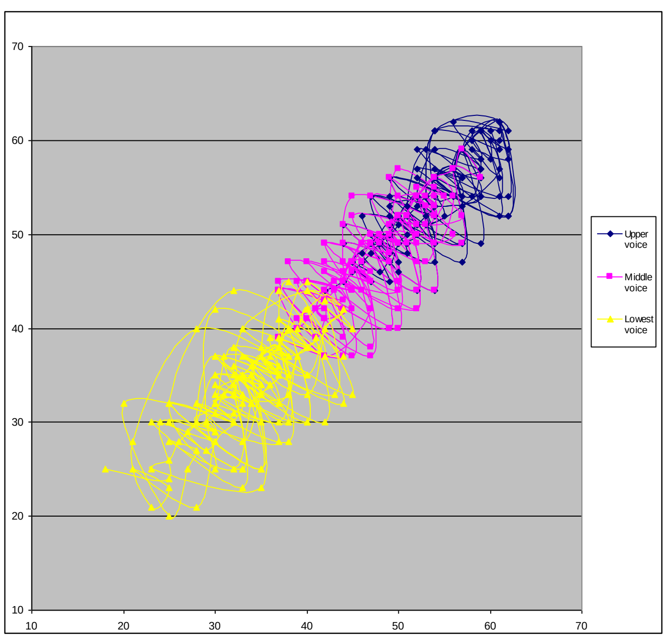
- Upper Voice: 1.1894
- Middle Voice: 1.2043
- Lower Voice: 1.2416

Dimension, Hsu

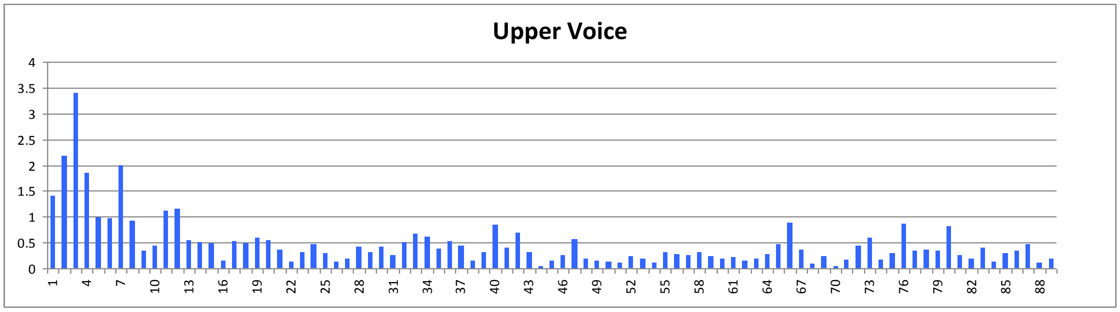
- Combined dimension: 1.2726

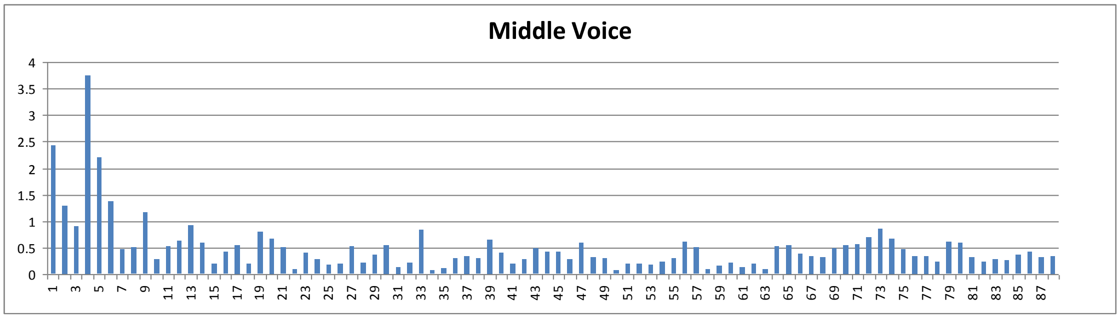


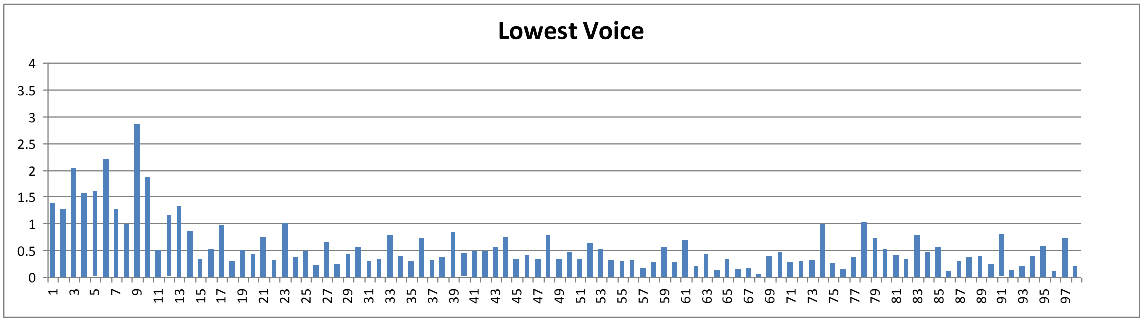
Attractor Plot



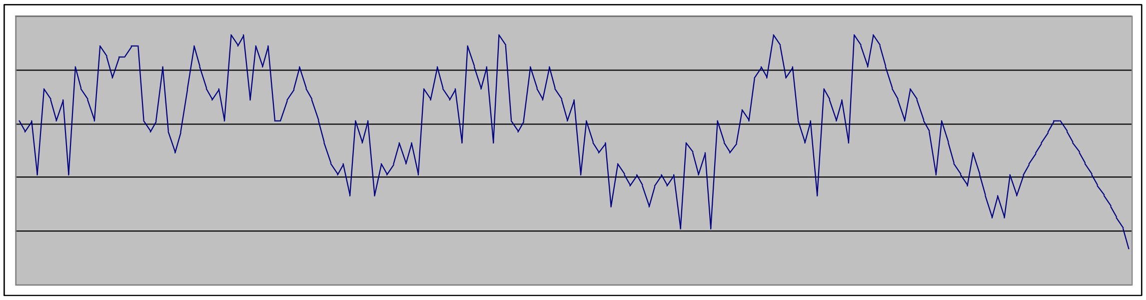
Spectral Analysis



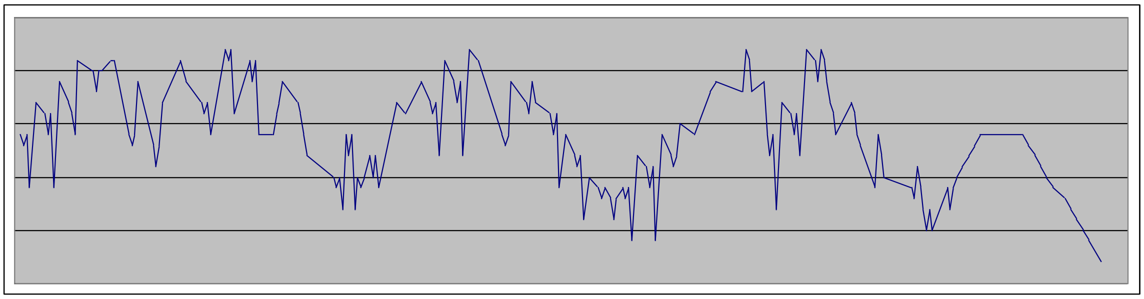




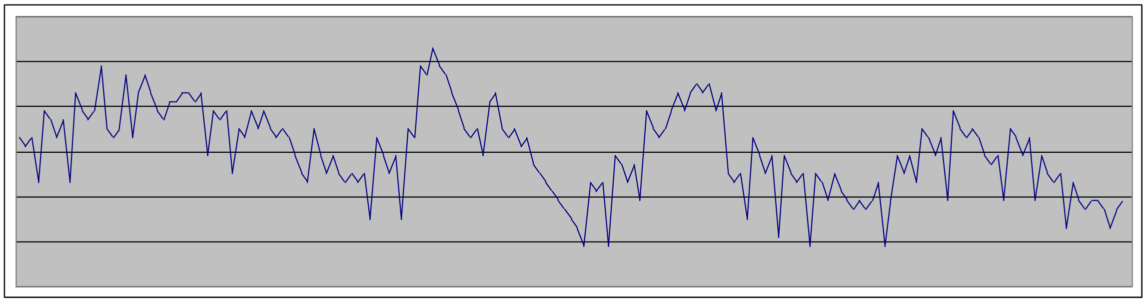
Inverse Transforms



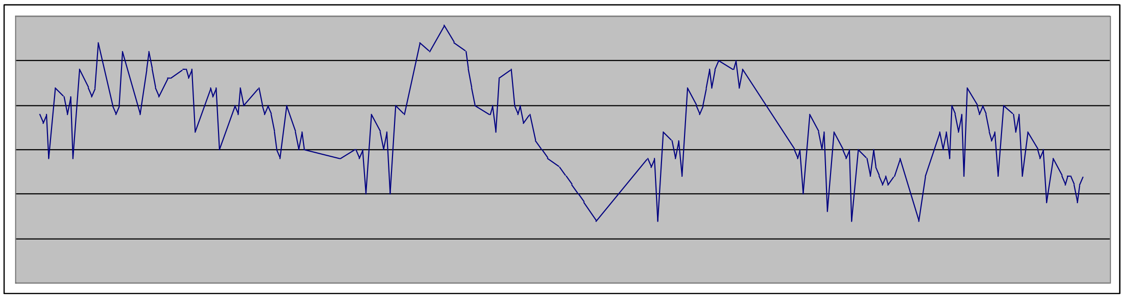
Inverse Transform (Upper Voice)



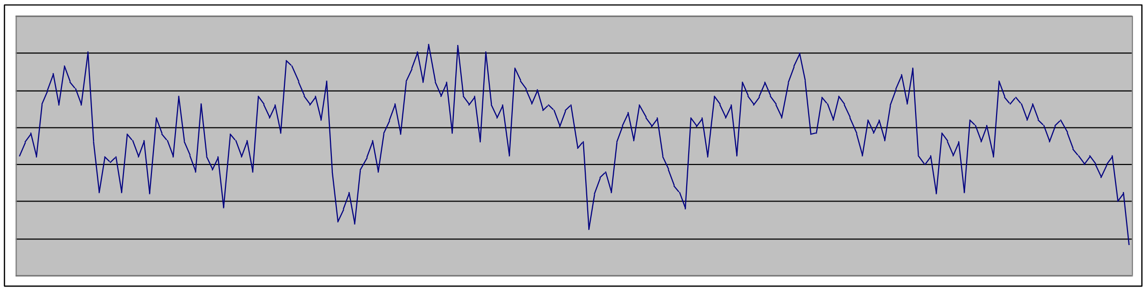
Original Graph (Upper Voice)



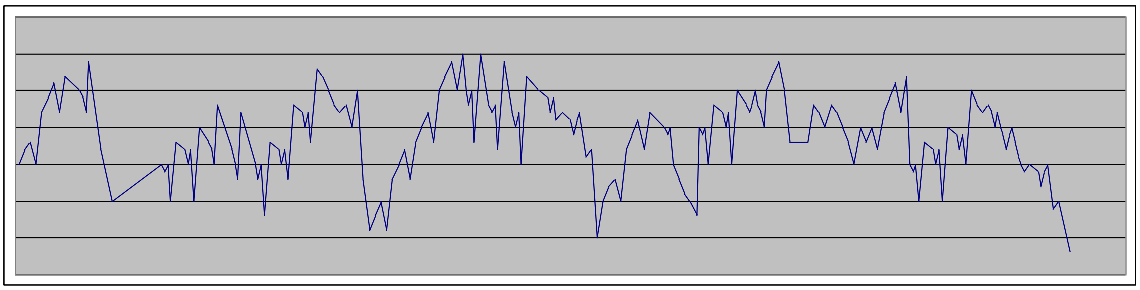
Inverse Transform (Middle Voice)



Original Graph (Middle Voice)



Inverse Transform (Lowest Voice)



Original Graph (Lowest Voice)

The attractor graph shows an even distribution of the orbits throughout all the voices except for a few wide orbits in the lowest voice. There is one strong basin of attraction in the lowest voice, and there are a few weak basins of attraction in the upper and middle voices.

All three of the harmonic spectrums for this piece have an unusual amount of energy through all of the upper harmonics. The upper and lowest voice spectrums do not have any unusually high amplitudes, but the spectrum for the middle voice has two high amplitudes. The harmonic spectrum of the lowest voice, while it does not have any extremely high amplitudes, has a high amount of energy throughout the entire spectrum.

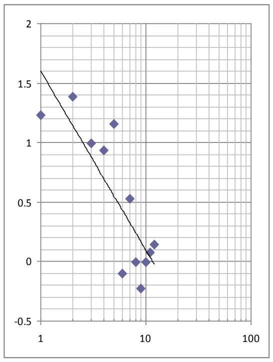
Three-Part Invention in Eb Major, No. 5, BWV 791 (1723)

Dimension, Madden

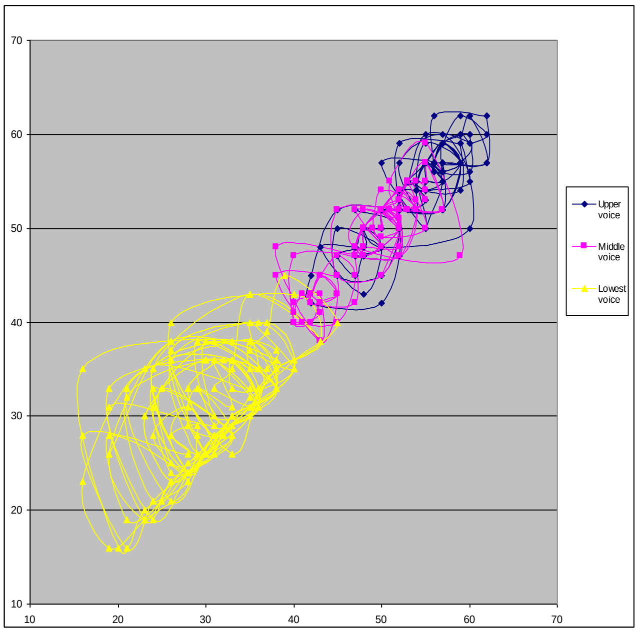
- Upper Voice: 1.1145
- Middle Voice: 1.1211
- Lower Voice: 1.2953

Dimension, Hsu

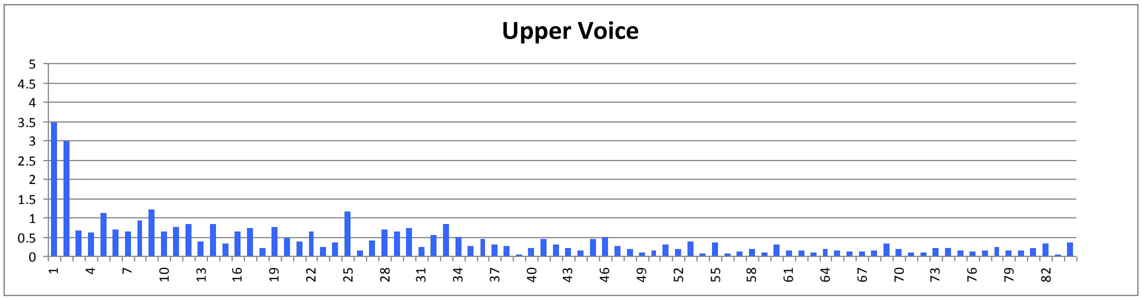
- Combined dimension: 1.5036

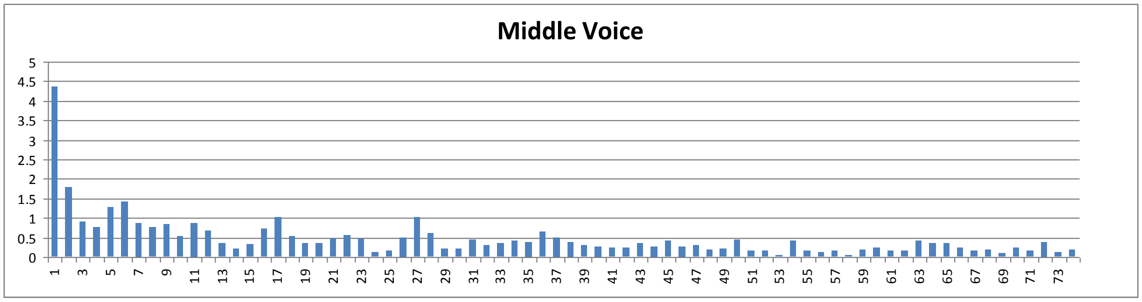


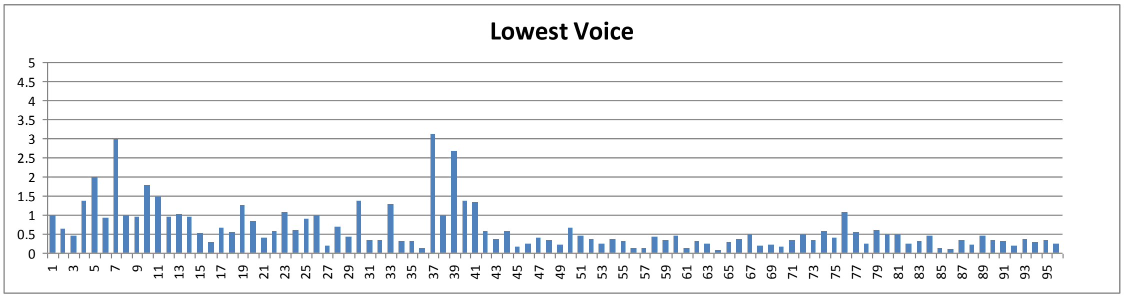
Attractor Plot



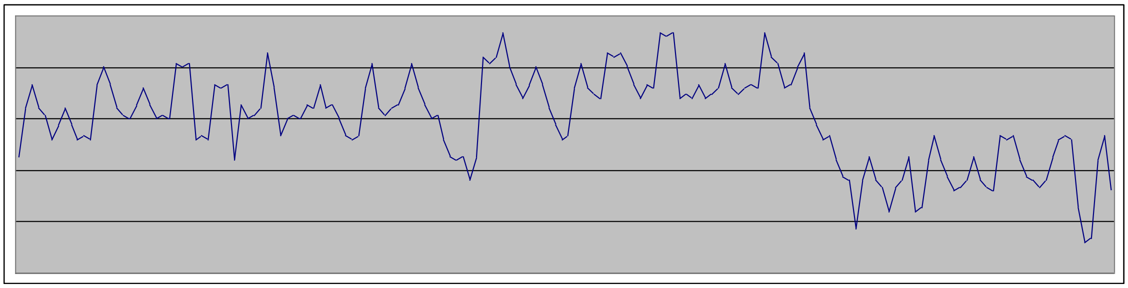
Spectral Analysis



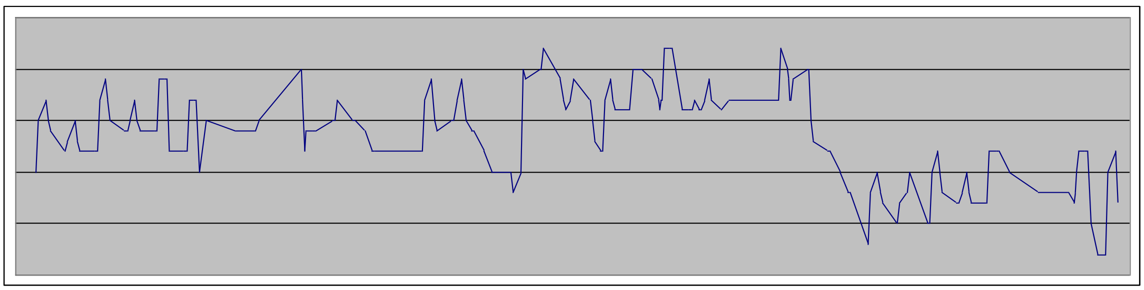




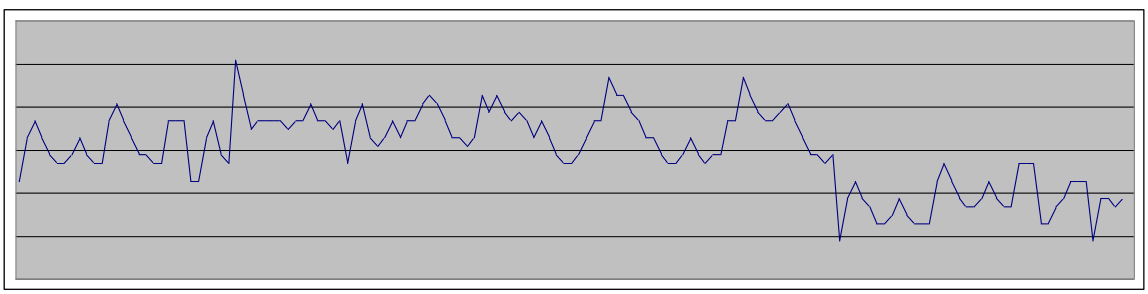
Inverse Transforms



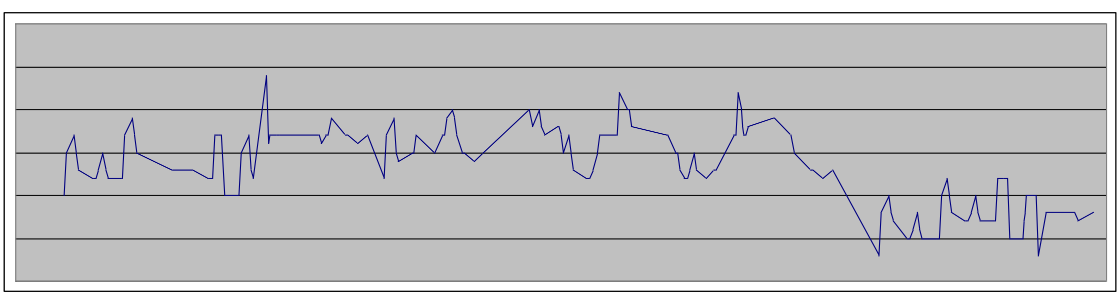
Inverse Transform (Upper Voice)



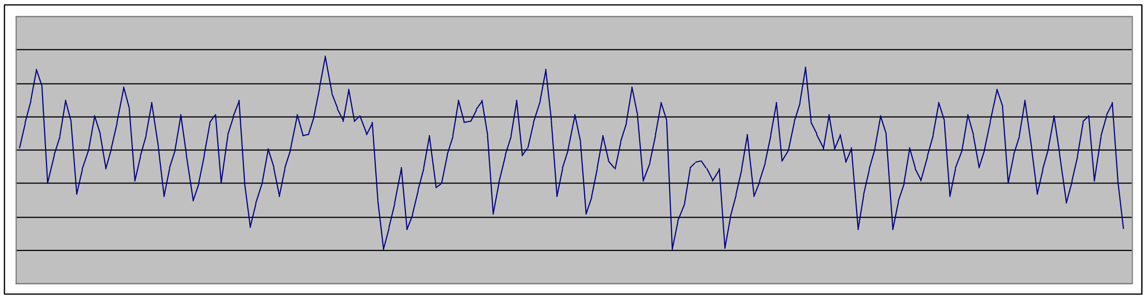
Original Graph (Upper Voice)



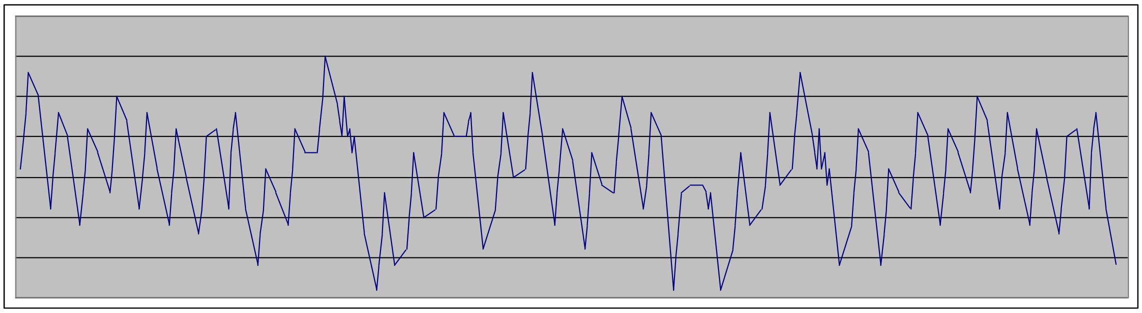
Inverse Transform (Middle Voice)



Original Graph (Middle Voice)



Inverse Transform (Lowest Voice)



Original Graph (Lowest Voice)

The attractor graph has an unusual appearance, especially in the lower voice. In the lowest voice, almost all of the intervals are ascending intervals, with only a few exceptions that are small descending intervals. There are two strong basins of attraction in the middle voice and one moderately strong basin of attraction in the upper voice, and there are no clear basins of attraction within the lowest voice.

The upper and middle voice harmonic spectrums begin two and one high amplitude harmonics, respectively. After these high amplitude harmonics, the harmonics gradually drop off and stabilize, and there is no significant activity in the upper harmonics.

The lowest voice harmonic spectrum begins with three low amplitude harmonics, after which the harmonic fluctuate between medium and high amplitudes through the eleventh harmonic, with a spike at the seventh harmonic. The amplitudes then appear to drop off and stabilize until a sudden spike in the harmonic amplitudes at about the thirty-seventh harmonic, which is followed by a low amplitude at the thirty-eighth harmonic and another high amplitude harmonic at the thirty-ninth harmonic. After this sudden activity, the amplitudes gradually drop off and stabilize with no more activity in the upper harmonics.

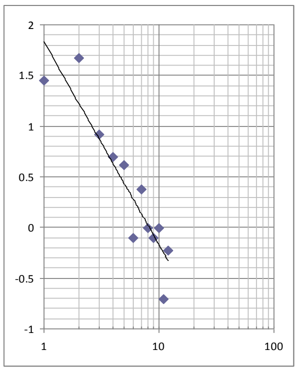
Three-Part Invention in E Major, No. 6, BWV 792 (1723)

Dimension, Madden

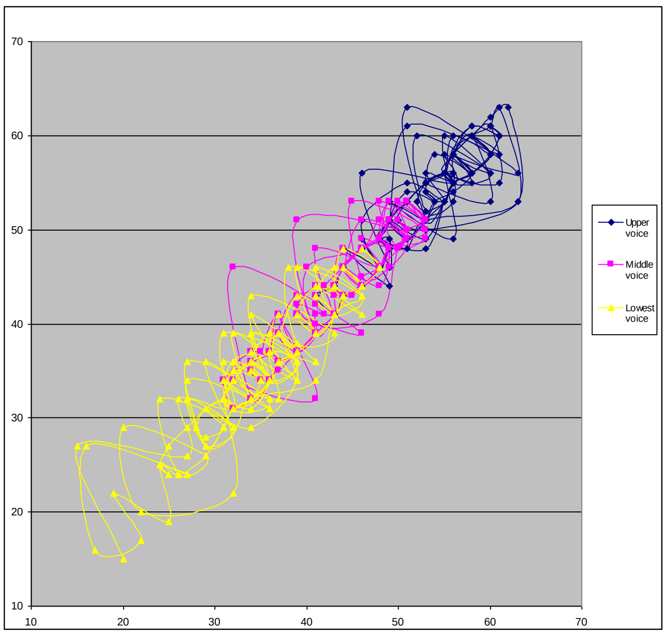
- Upper Voice: 1.1651
- Middle Voice: 1.1571
- Lower Voice: 1.1982

Dimension, Hsu

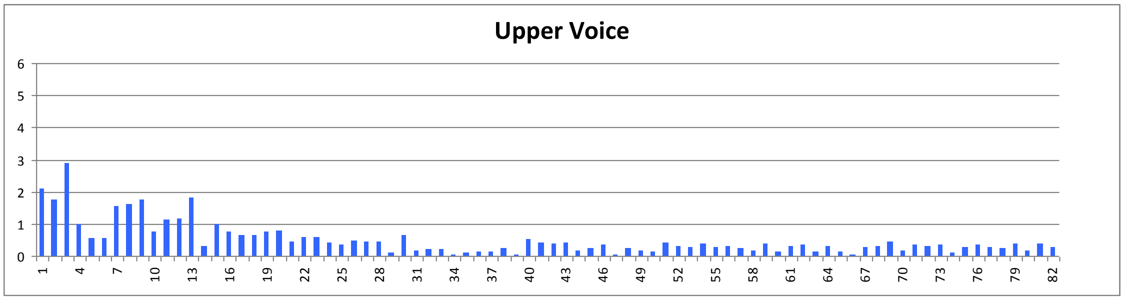
- Combined dimension: 2.0007

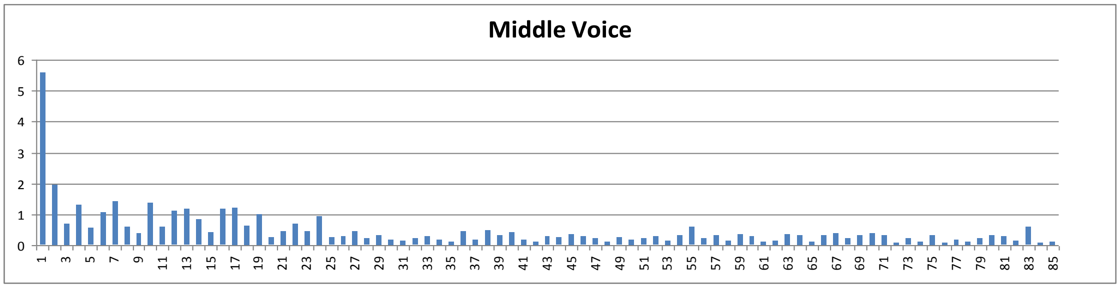


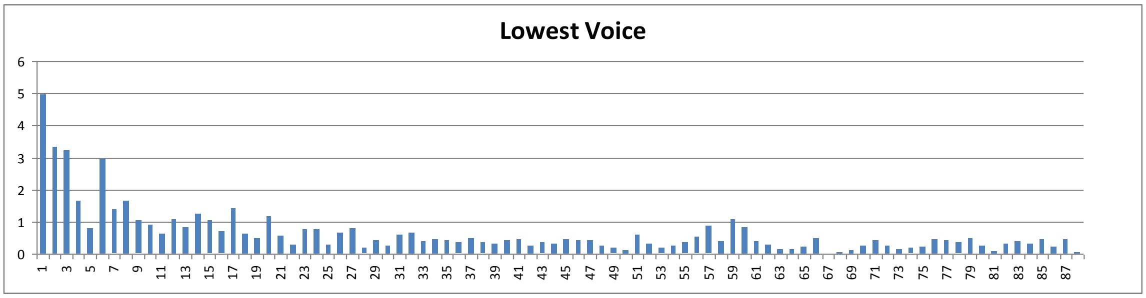
Attractor Plot



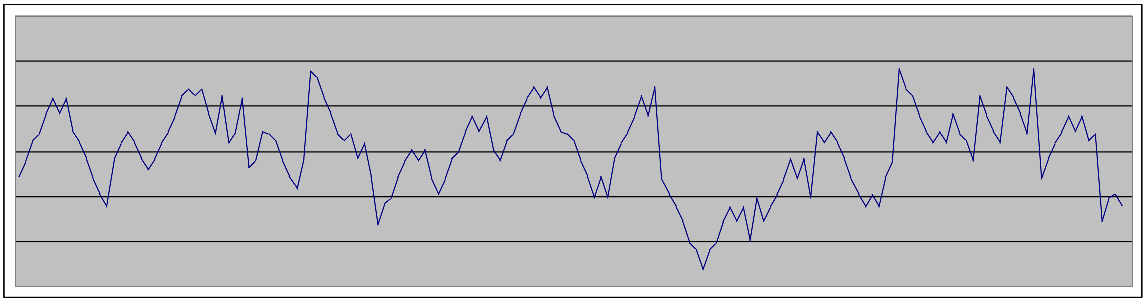
Spectral Analysis



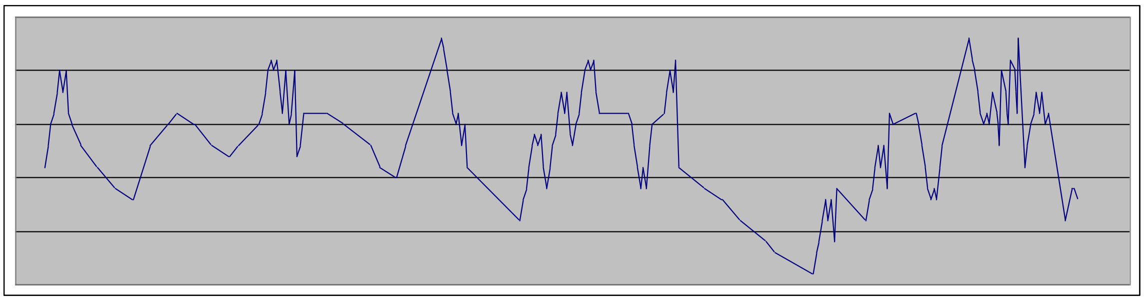




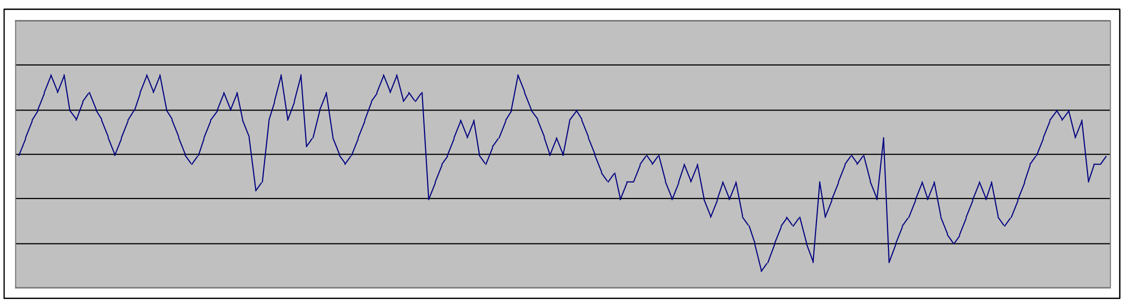
Inverse Transforms



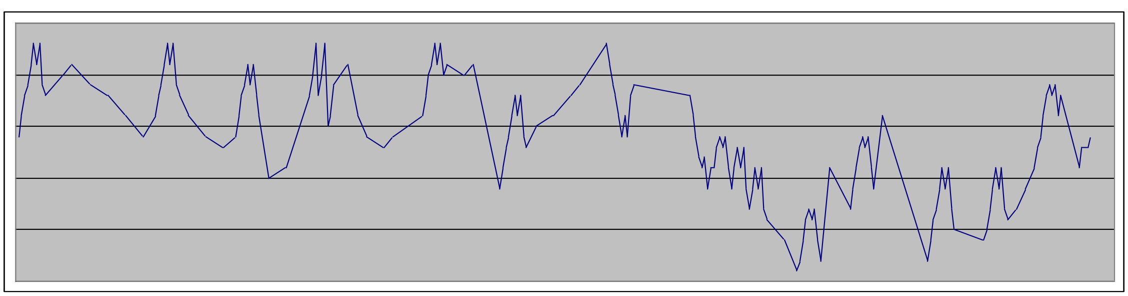
Inverse Transform (Upper Voice)



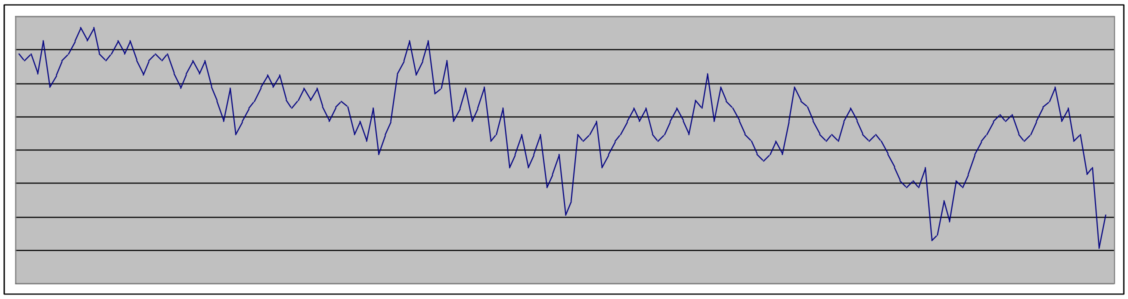
Original Graph (Upper Voice)



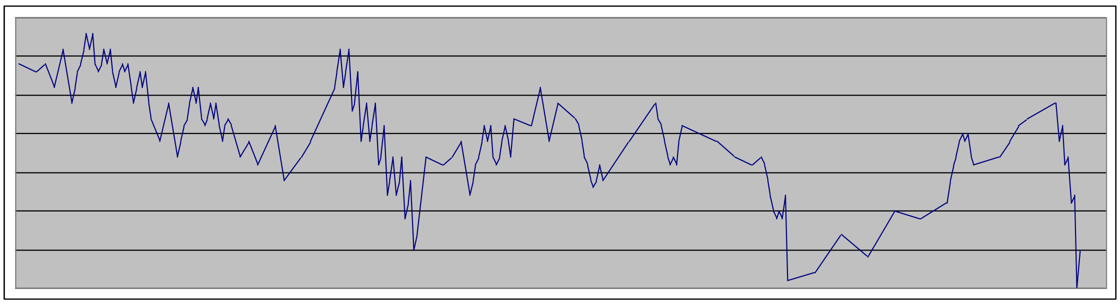
Inverse Transform (Middle Voice)



Original Graph (Middle Voice)



Inverse Transform (Lowest Voice)



Original Graph (Lowest Voice)

The Hsu dimension on this piece is once again very high, apparently due to the scalar nature of the motive.

The attractor graph does not have any unusual characteristics except that the ranges of the voices overlap more than is typical in most of the pieces. There is one strong basin of attraction in the upper voice and the middle voice. Although there are a few places where the notes seem to form small clusters, there are no strong basins of attraction in the lowest voice.

The harmonic spectrums for this piece have no unusual distributions. The middle and lowest voice spectrums each have one high amplitude, after which the harmonics gradually drop off. The upper voice spectrum does not have any unusually high amplitudes, and it also drops off and stabilizes relatively early in the spectrum.

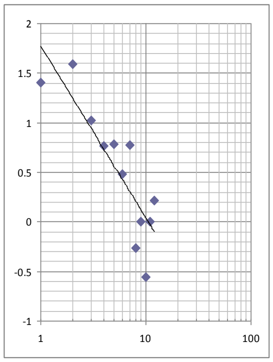
Three-Part Invention in E Minor, No. 7, BWV 793 (1723)

Dimension, Madden

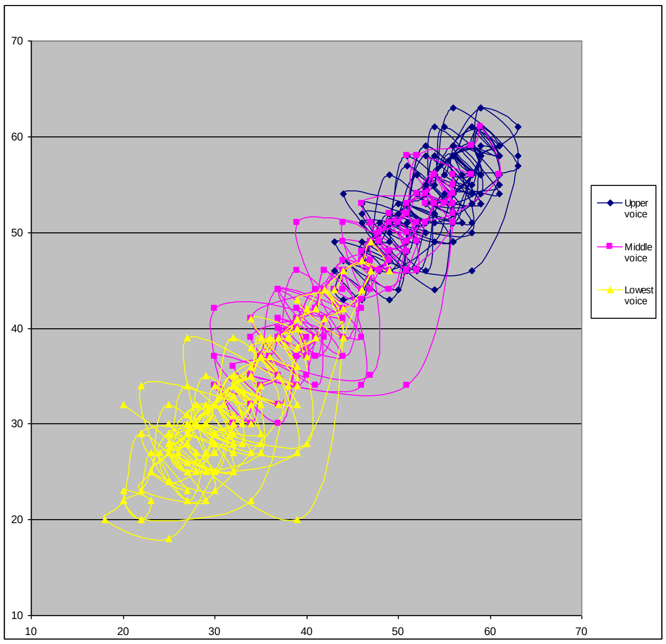
- Upper Voice: 1.1720
- Middle Voice: 1.1899
- Lower Voice: 1.2000

Dimension, Hsu

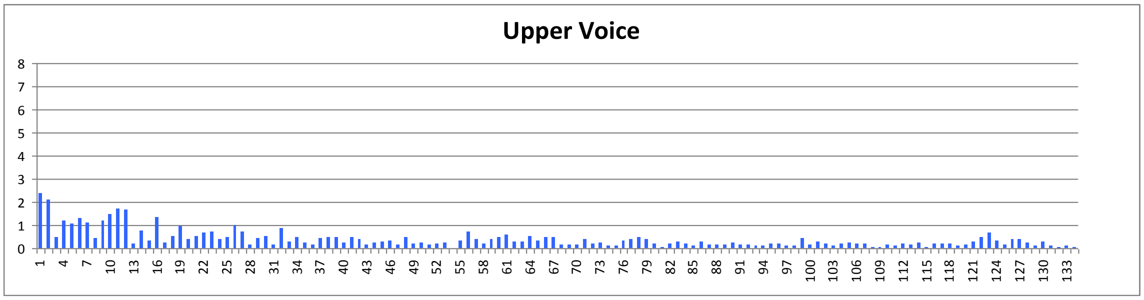
- Combined dimension: 1.7352

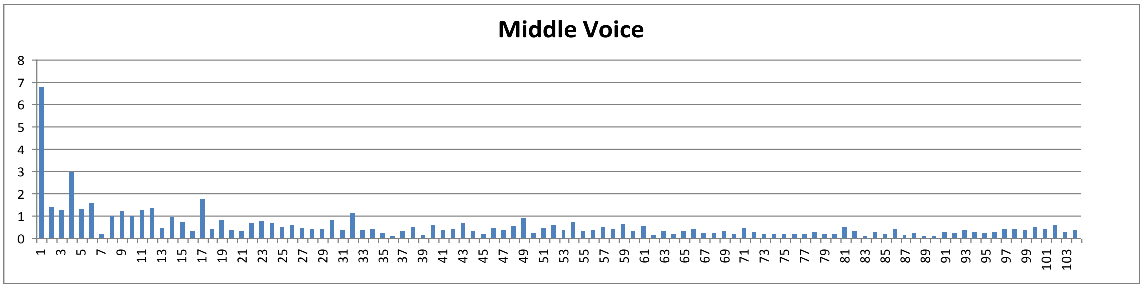


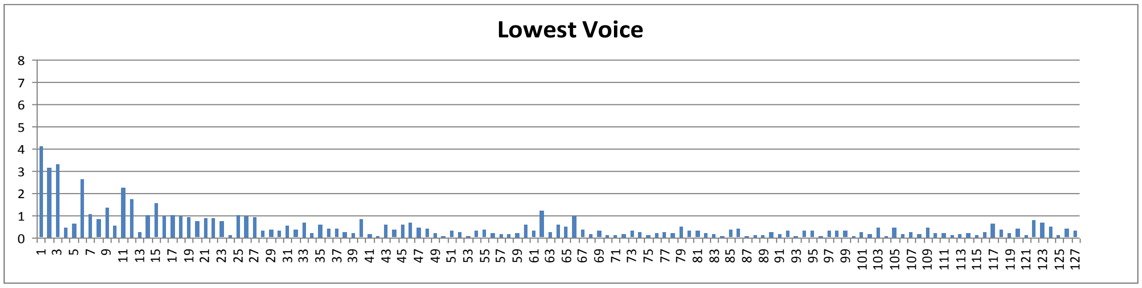
Attractor Plot



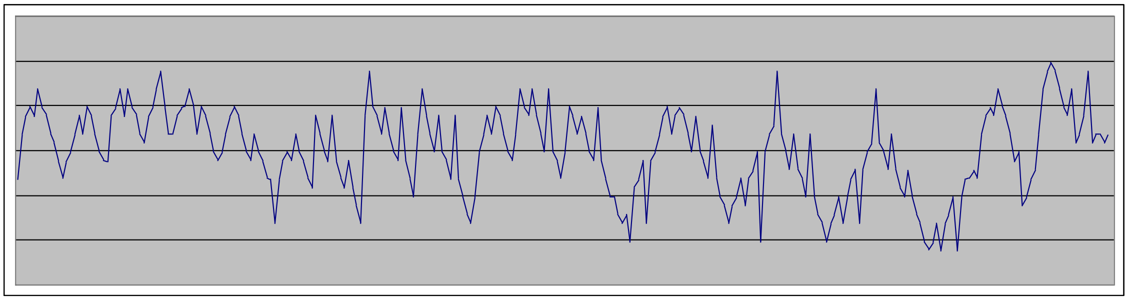
Spectral Analysis



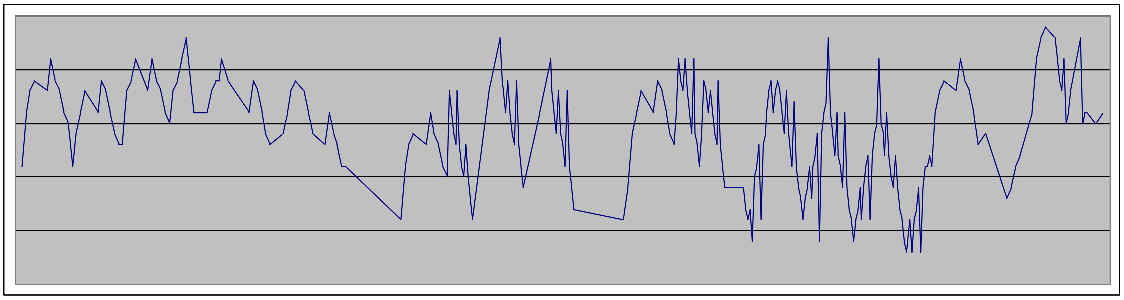




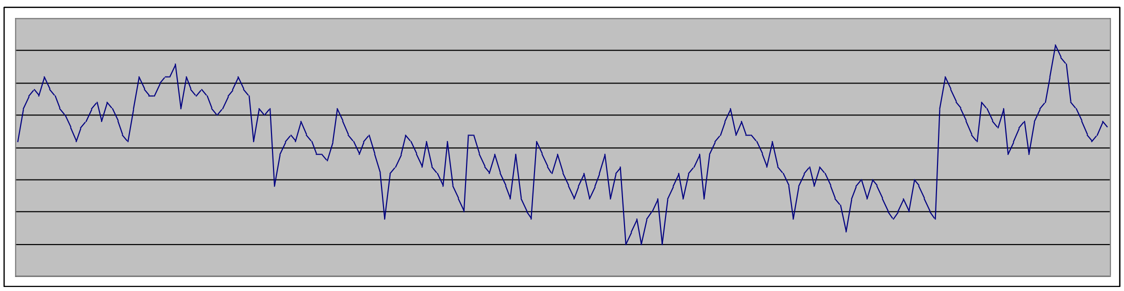
Inverse Transforms



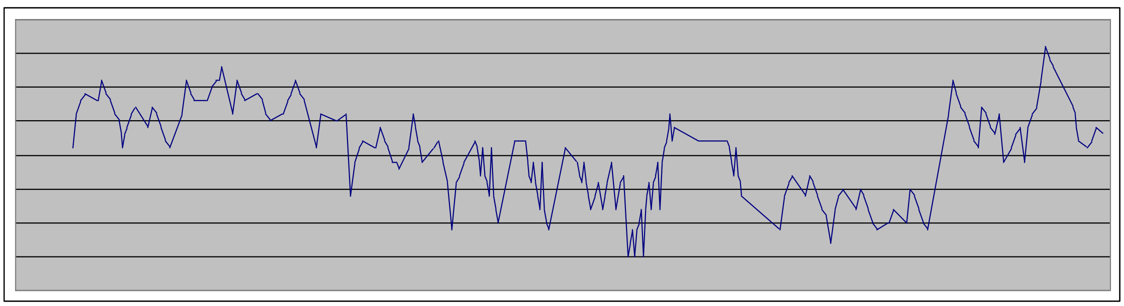
Inverse Transform (Upper Voice)



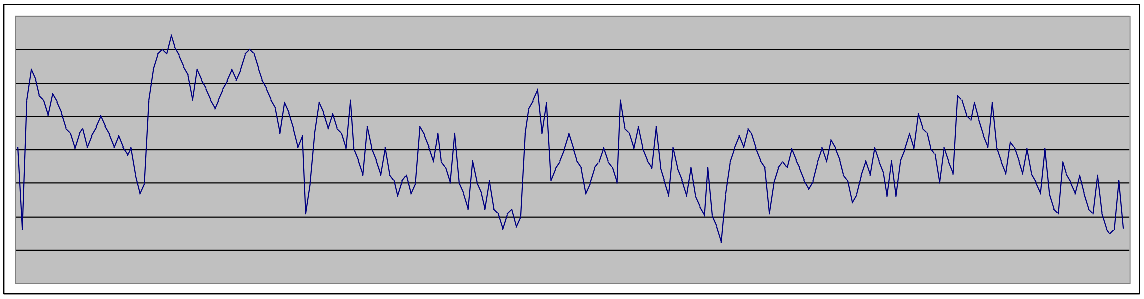
Original Graph (Upper Voice)



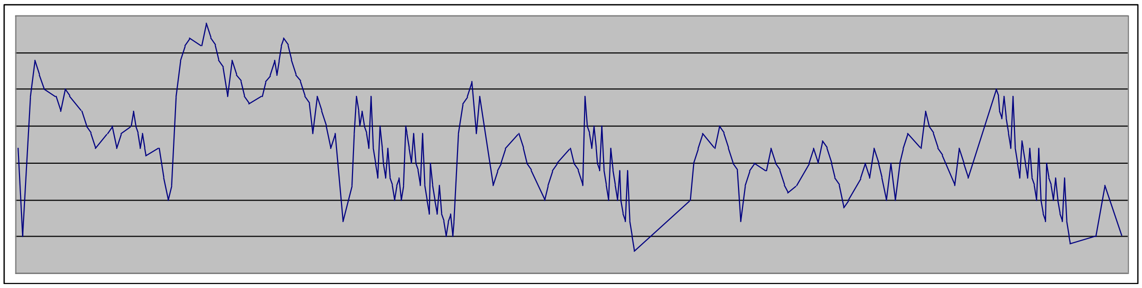
Inverse Transform (Middle Voice)



Original Graph (Middle Voice)



Inverse Transform (Lowest Voice)



Original Graph (Lowest Voice)

The attractor graph does not demonstrate any unusual behavior, although there are more wide orbits caused by large intervals in the lowest and middle voices than in the upper voice. There is a strong basin of attraction in the lowest voice, and there are two weak basins of attraction shared by the upper and middle voices.

There is nothing unusual in the harmonic spectrums except for a slight increase in amplitude at about the sixty-first harmonic of the lowest voice.

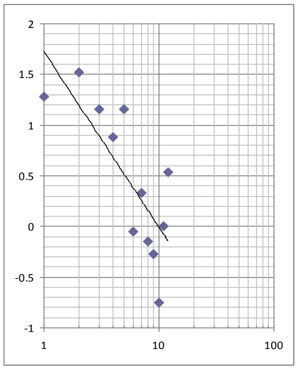
Three-Part Invention in F Major, No. 8, BWV 794 (1723)

Dimension, Madden

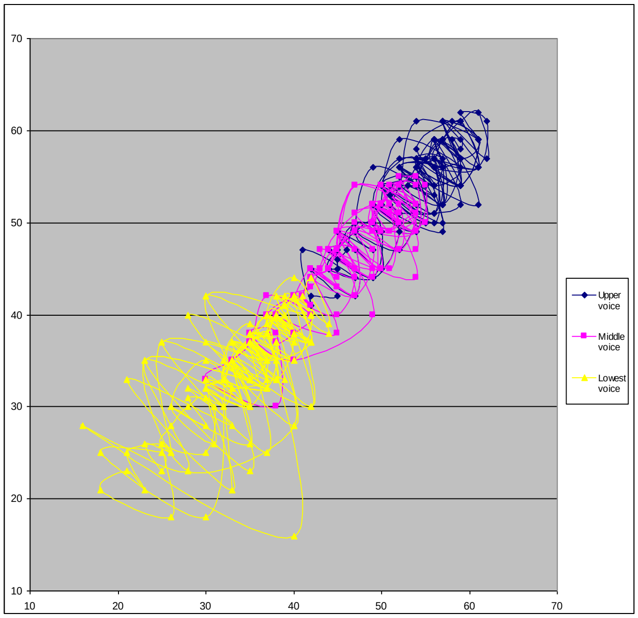
- Upper Voice: 1.1923
- Middle Voice: 1.2035
- Lower Voice: 1.2328

Dimension, Hsu

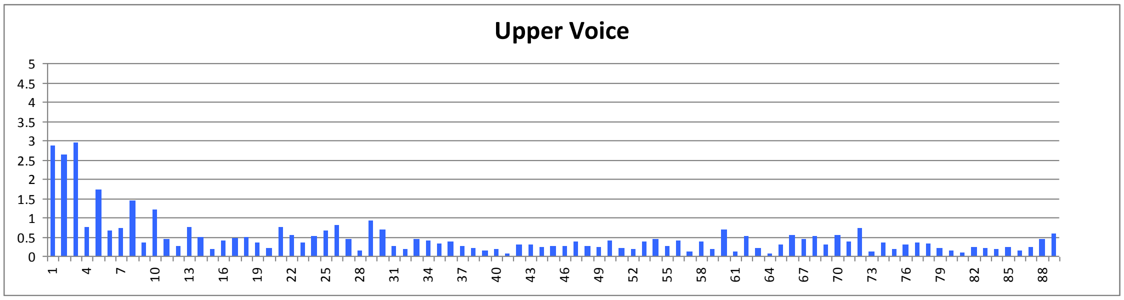
- Combined dimension: 1.7339

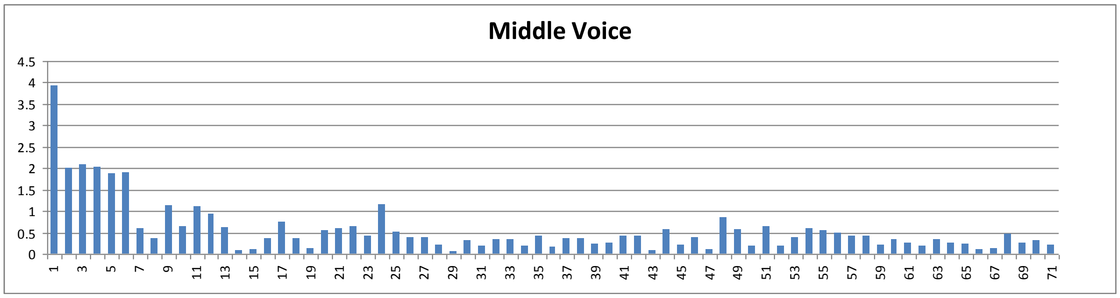


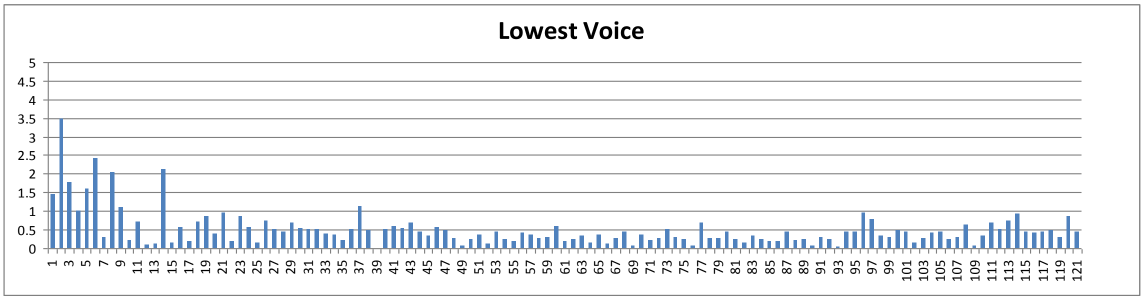
Attractor Plot



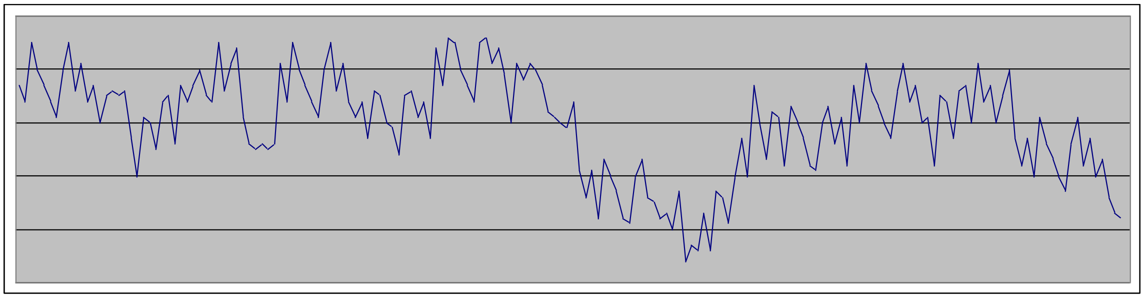
Spectral Analysis



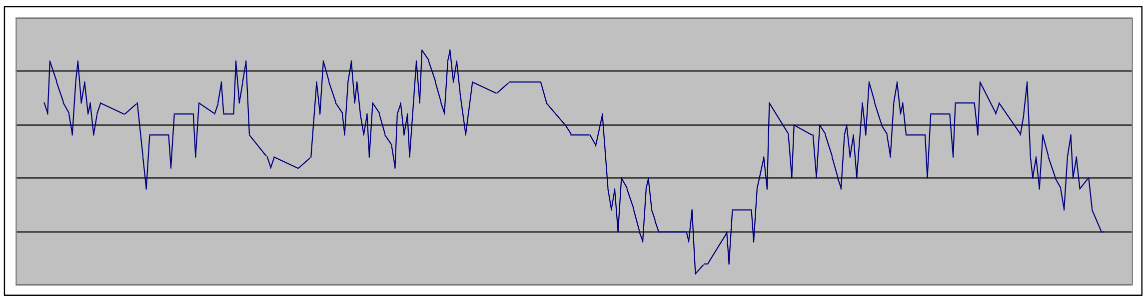




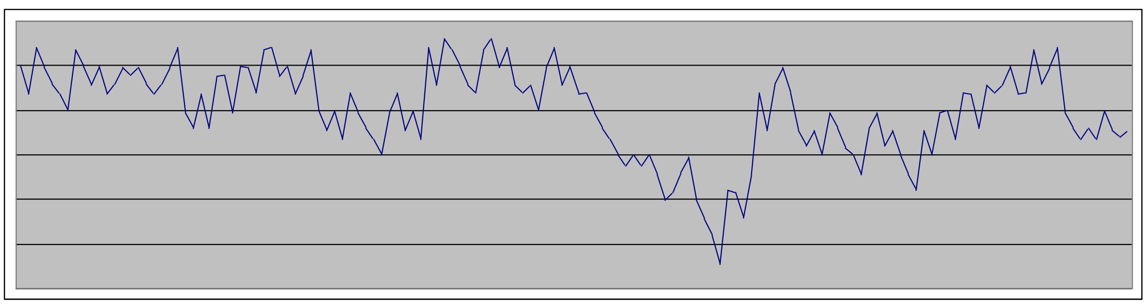
Inverse Transforms



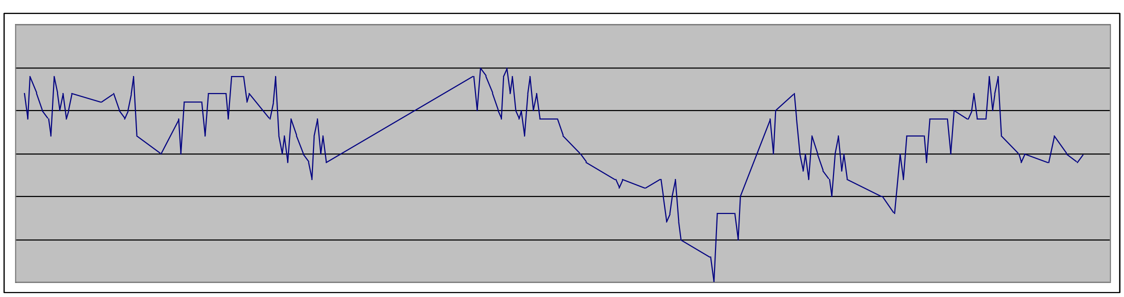
Inverse Transform (Upper Voice)



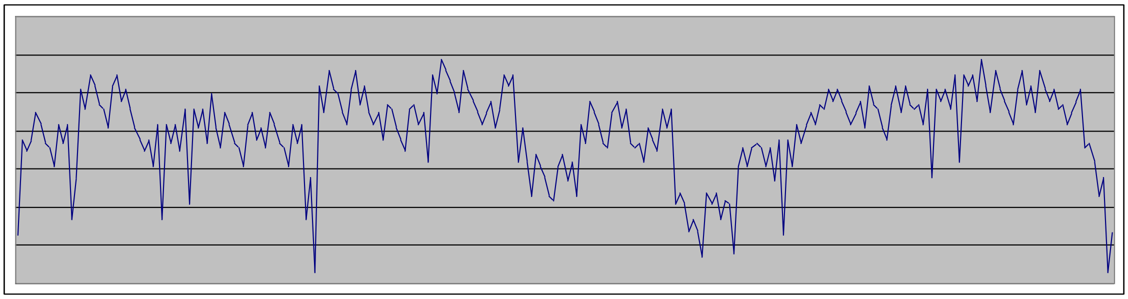
Original Graph (Upper Voice)



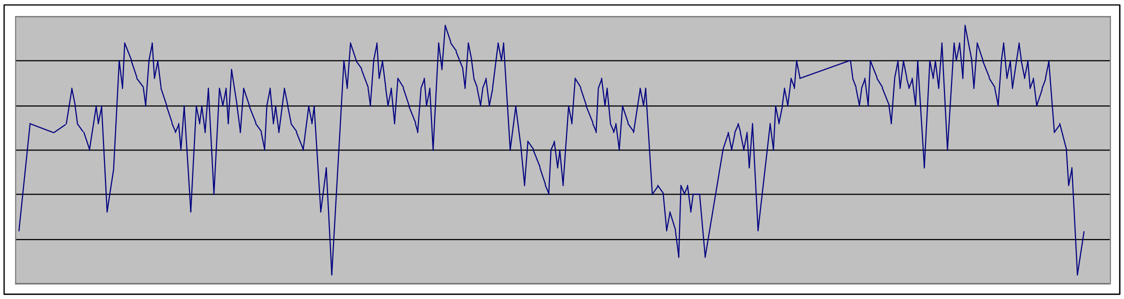
Inverse Transform (Middle Voice)



Original Graph (Middle Voice)



Inverse Transform (Lowest Voice)



Original Graph (Lowest Voice)

The Hsu dimension for this piece is slightly high. Although the motive for this piece is not truly scalar in nature, it contains some use of scales, which is the likely cause of the high dimension.

The wide orbits in the attractor graph show that the larger intervals are primarily in the lowest voice. There are several strong basins of attraction spread throughout the graph.

The harmonic spectrum of the upper voice has no unusual characteristics. The middle and lowest voices both have fluctuating high and low amplitudes in the beginning harmonics that are followed by extremely low amplitudes, after which the harmonics appear to stabilize slightly, although energy continues throughout the upper harmonics in both voices. The low voice has a slight but significant increase in energy at the end of its harmonic spectrum.

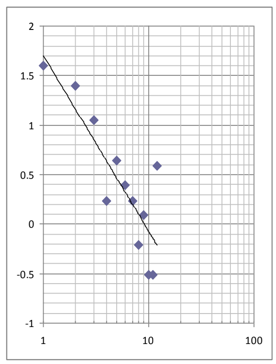
Three-Part Invention in F Minor, No. 9, BWV 795 (1723)

Dimension, Madden

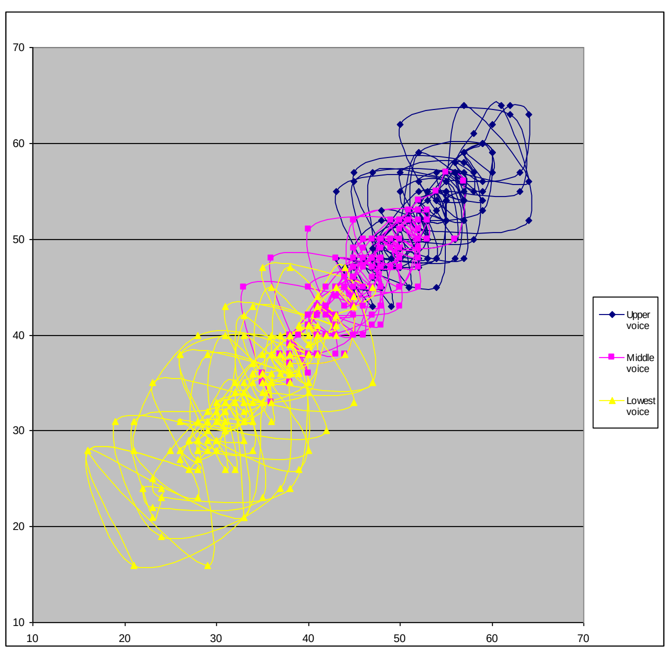
- Upper Voice: 1.1669
- Middle Voice: 1.1346
- Lower Voice: 1.2099

Dimension, Hsu

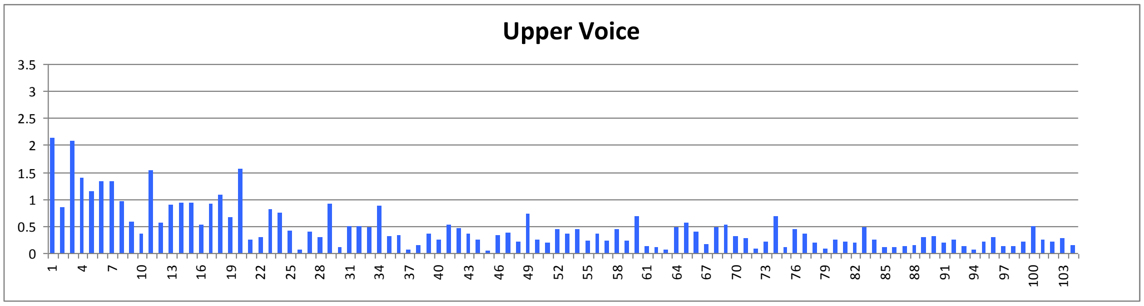
- Combined dimension: 1.7767

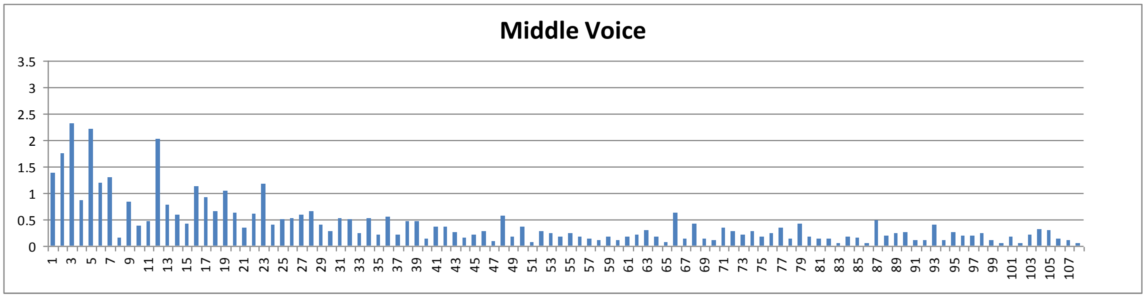


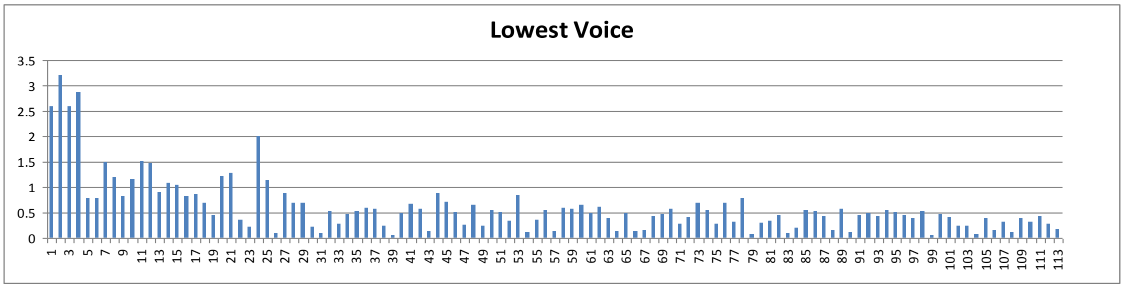
Attractor Plot



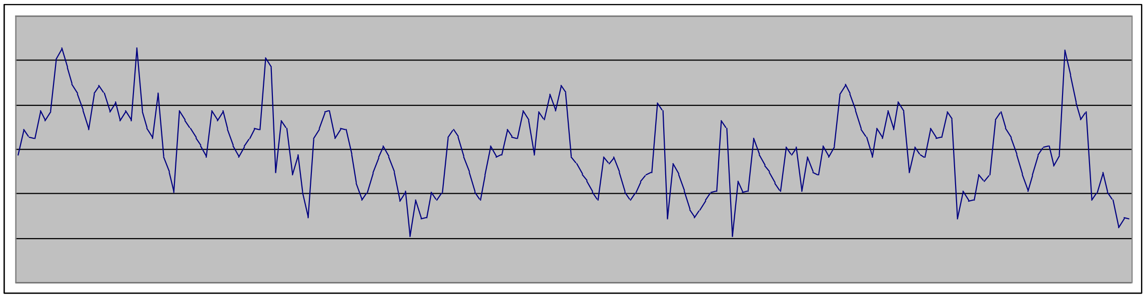
Spectral Analysis



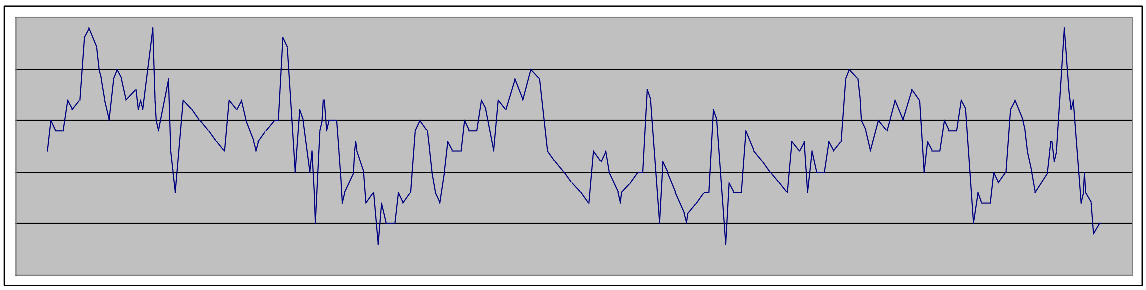




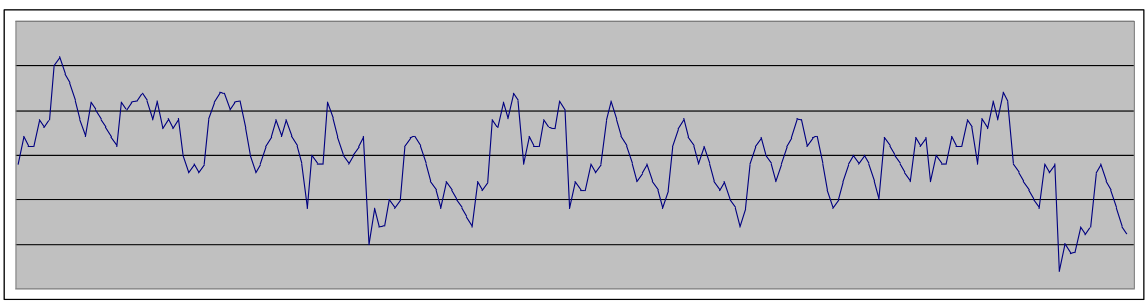
Inverse Transforms



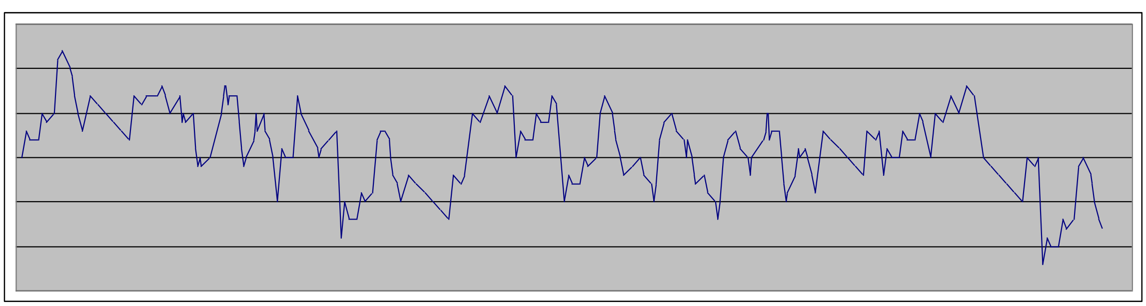
Inverse Transform (Upper Voice)



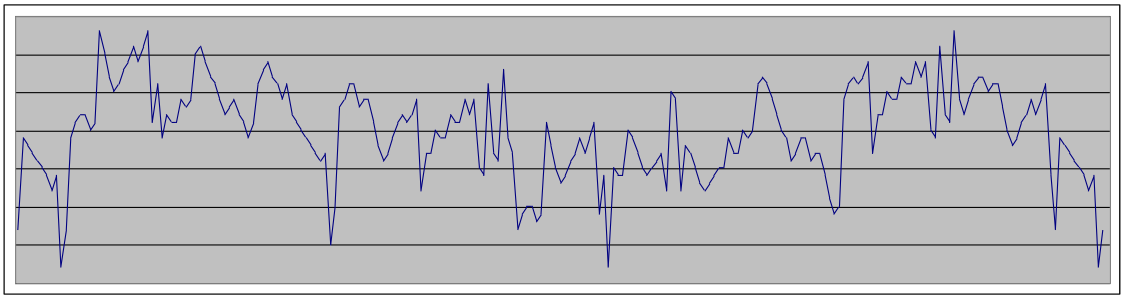
Original Graph (Upper Voice)



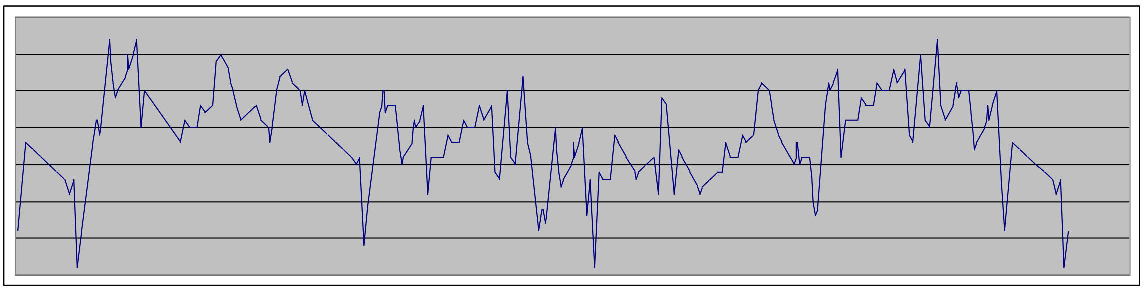
Inverse Transform (Middle Voice)



Original Graph (Middle Voice)



Inverse Transform (Lowest Voice)



Original Graph (Lowest Voice)

The Hsu dimension for this piece is unusually high, probably due to the scalar tendencies of the motive. Although the motive is interrupted by rests, the notes are still scalar in nature.

The attractor graph shows a relatively even distribution throughout the range of all the voices, and although there are more large intervals in the lowest voice, the wide orbits in the middle and upper voices show the presence of large intervals within these voices. The notes seem to cluster fairly evenly all along the diagonal line, disguising the presence of any strange attractors if there are any present.

The harmonic spectrum of the upper voice does not have any remarkably high amplitudes. However, the amplitudes begin at a medium height and do not really drop off until about the thirty-fifth harmonic, after which the amplitudes drop off and become fairly stable, although there are a few slight increases in energy in the upper harmonics.

The harmonic spectrum of the middle voice also has no high amplitudes, and the amplitudes do not drop off until about the twenty-third harmonic. Although the upper harmonics for the middle voice have less energy than that of the upper voice, there is still a small amount of energy throughout the upper harmonics.

The harmonic spectrum of the lowest voice begins with four high amplitude harmonics, after which the amplitudes drop to a medium height, and after a spike at about the twenty-fourth harmonic, the amplitudes gradually stabilize somewhat. There is a significant amount of energy through about the seventy-seventh harmonic, after which the amplitudes drop slightly for the remainder of the spectrum.

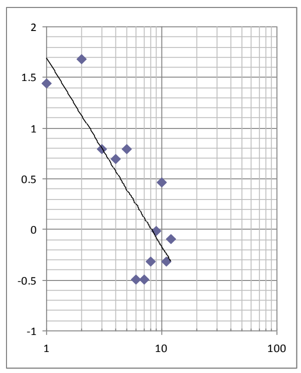
Three-Part Invention in G Major, No. 10, BWV 796 (1723)

Dimension, Madden

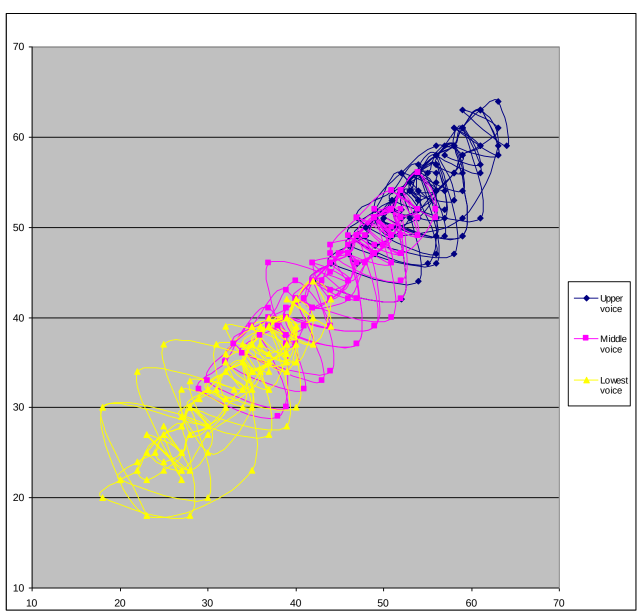
- Upper Voice: 1.1606
- Middle Voice: 1.1785
- Lower Voice: 1.1864

Dimension, Hsu

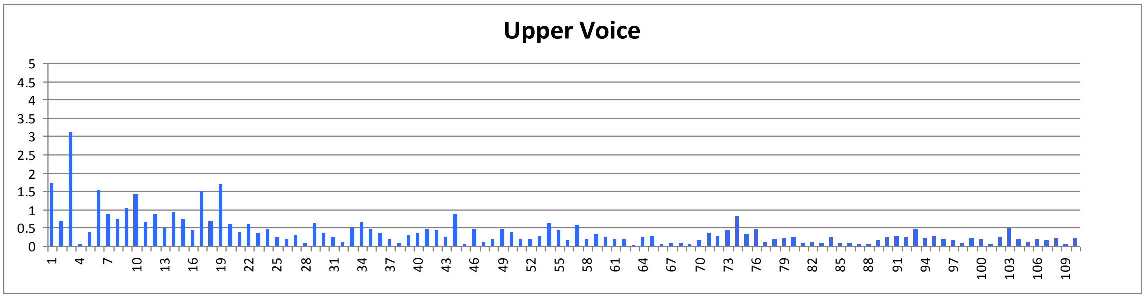
- Combined dimension: 1.8623

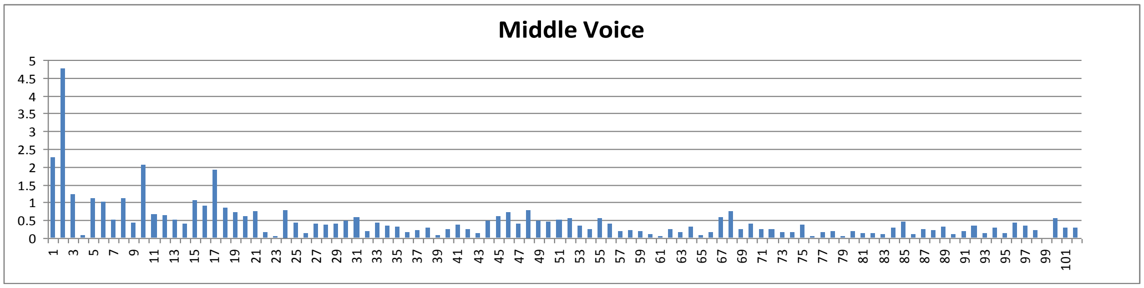


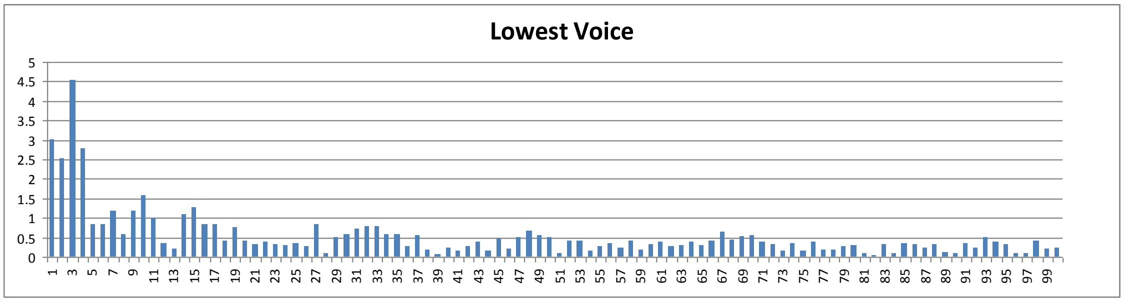
Attractor Plot



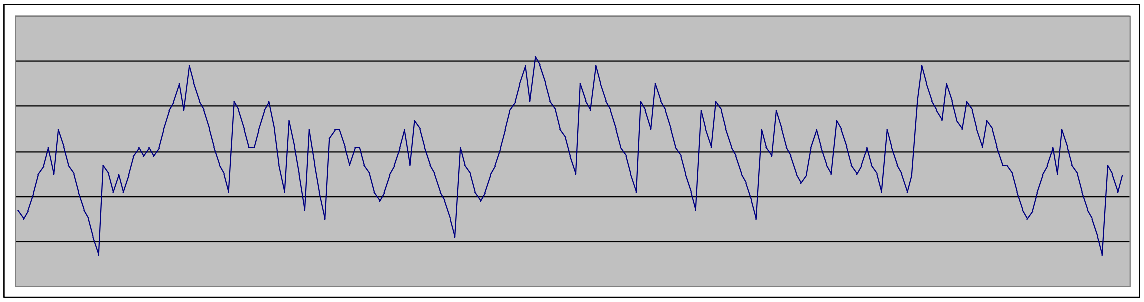
Spectral Analysis



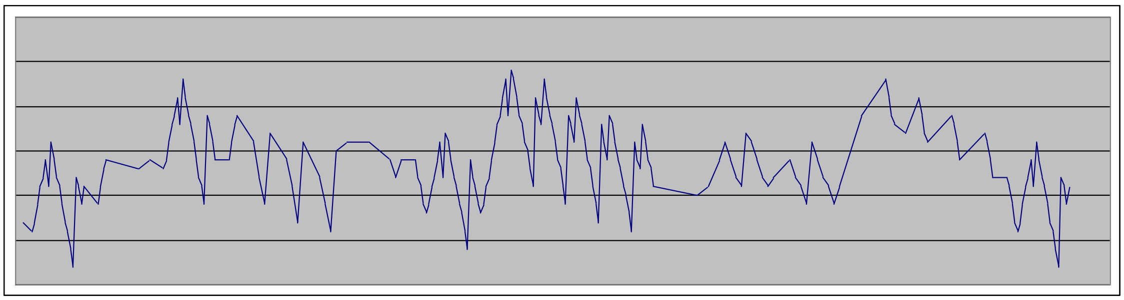




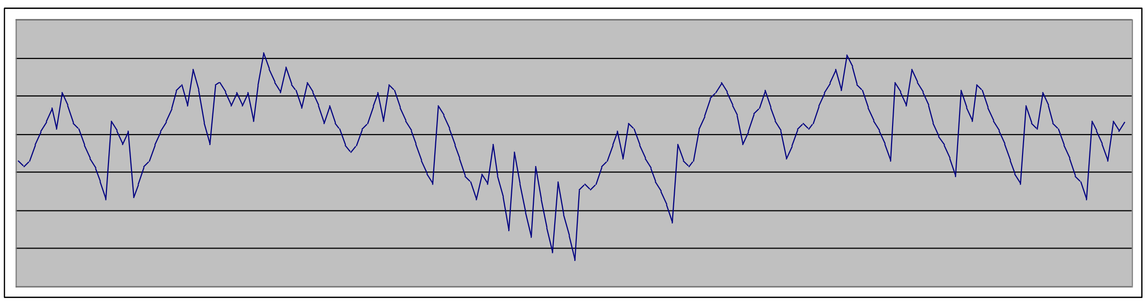
Inverse Transforms



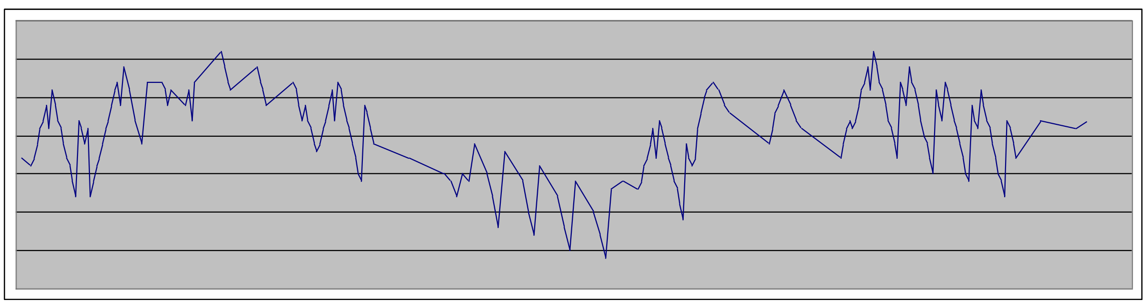
Inverse Transform (Upper Voice)



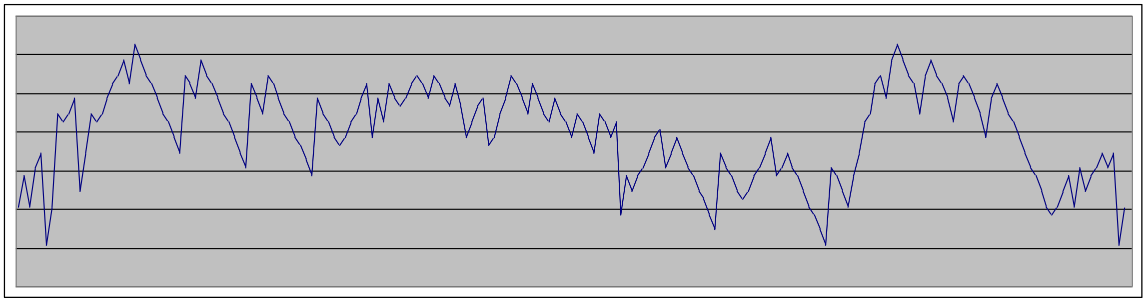
Original Graph (Upper Voice)



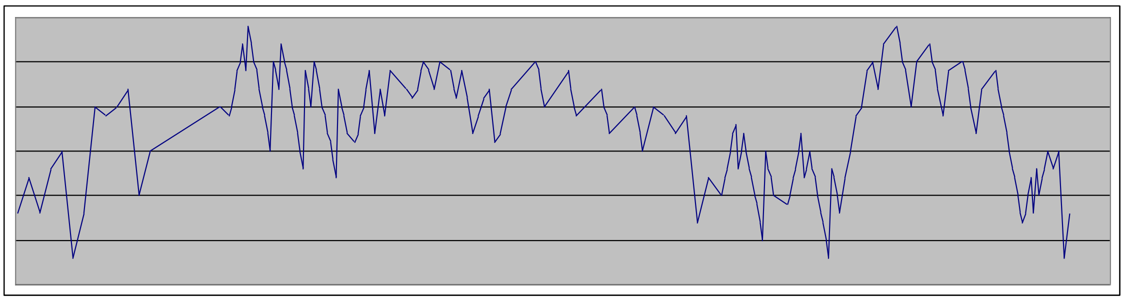
Inverse Transform (Middle Voice)



Original Graph (Middle Voice)



Inverse Transform (Lowest Voice)



Original Graph (Lowest Voice)

This three-part invention also has a motive of a predominantly scalar nature, which accounts for the unusually high dimension computed by the Hsus' method.

The attractor graph shows a predominance of descending intervals (large orbits to the right side of the diagonal line), although there are a few large ascending intervals. There is a strong basin of attraction shared by the lowest and middle voices, and a strong basin of attraction in each of the middle and upper voices.

The harmonic spectrum of the upper voice does not have any unusually high amplitudes; instead, there is an unusually high number of extremely low amplitudes scattered throughout the spectrum. The amplitudes drop off at about the twentieth harmonic and remain stable except for very small spikes at about the forty-fourth and seventy-fifth harmonic.

The harmonic spectrum of the middle voice has an unusually high amplitude at the second harmonic, after which the harmonics gradually stabilize with no more significant activity after the seventeenth harmonic.

The harmonic spectrum of the lowest voice begins with four high amplitude harmonics, after which the amplitudes gradually stabilize. There is no significant activity in the upper harmonics.

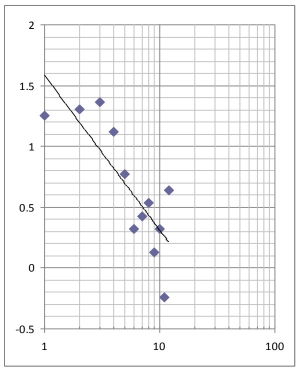
Three-Part Invention in G Minor, No. 11, BWV 797 (1723)

Dimension, Madden

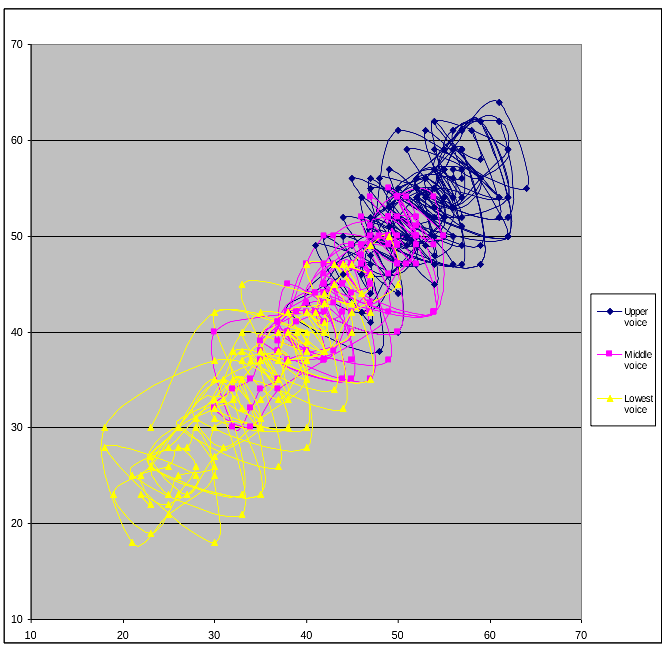
- Upper Voice: 1.2433
- Middle Voice: 1.2394
- Lower Voice: 1.2610

Dimension, Hsu

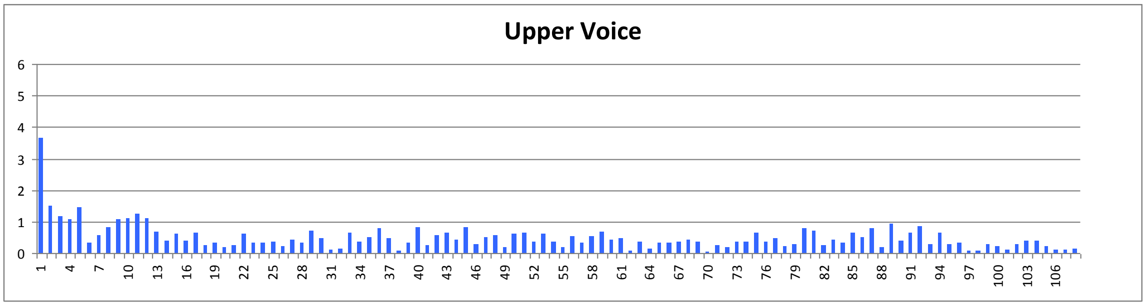
- Combined dimension: 1.2761

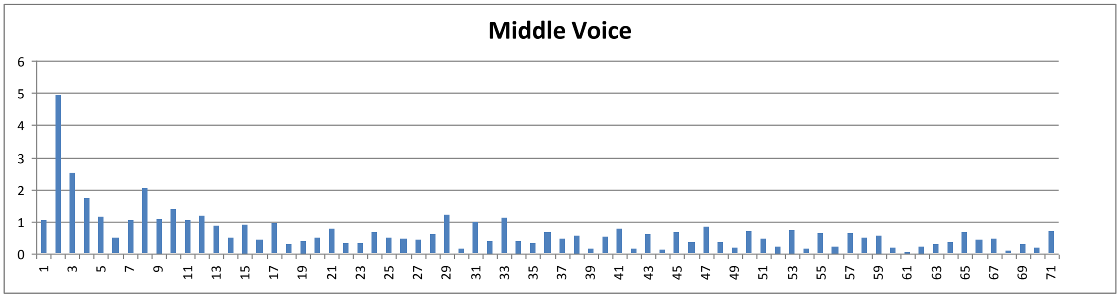


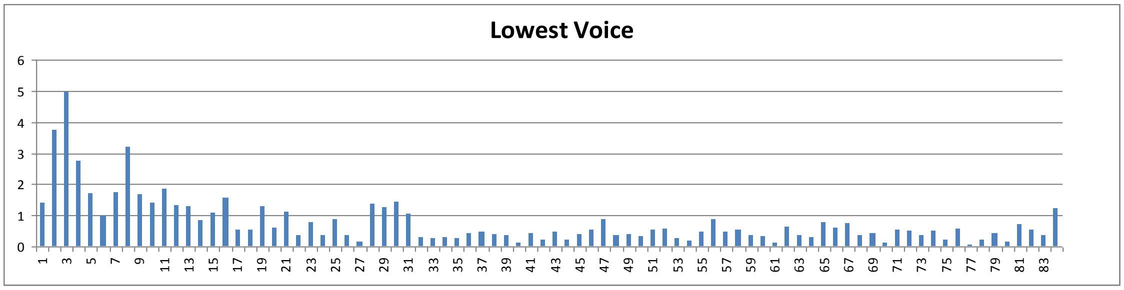
Attractor Plot



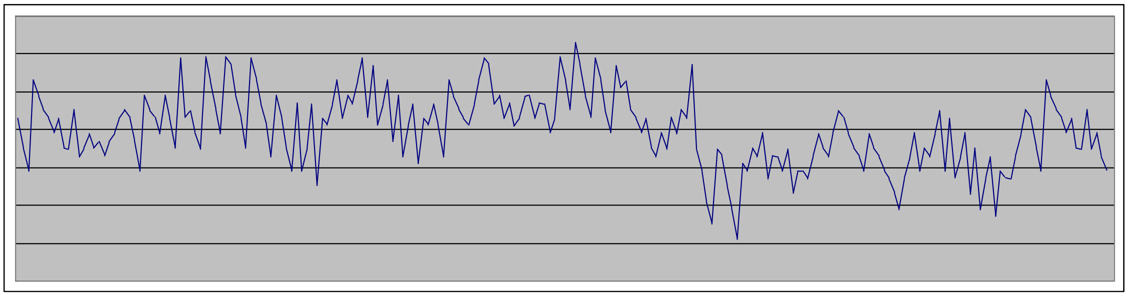
Spectral Analysis



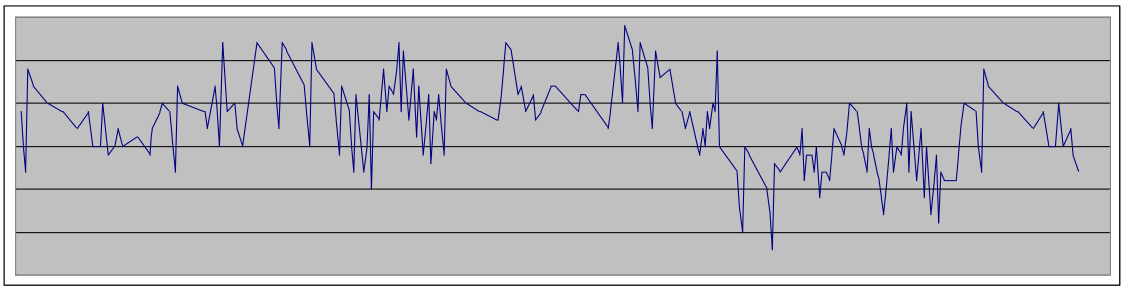




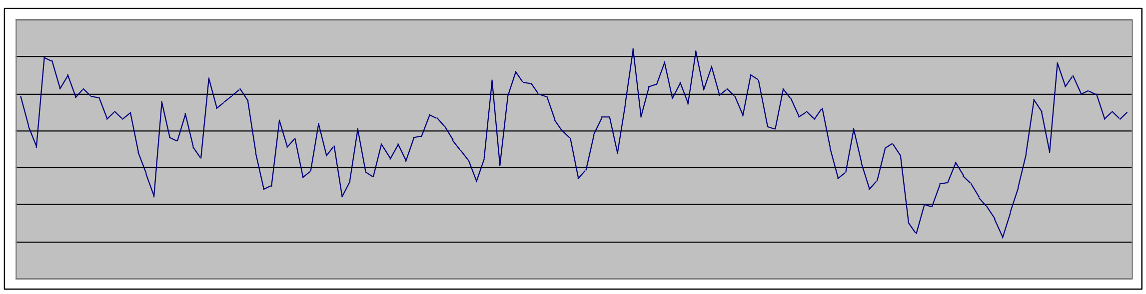
Inverse Transforms



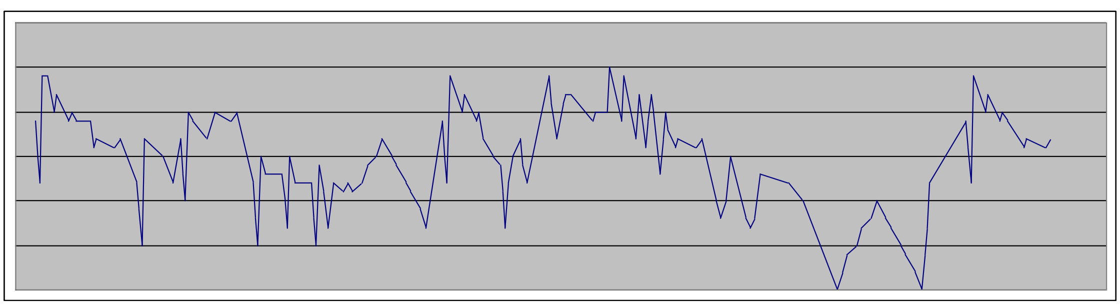
Inverse Transform (Upper Voice)



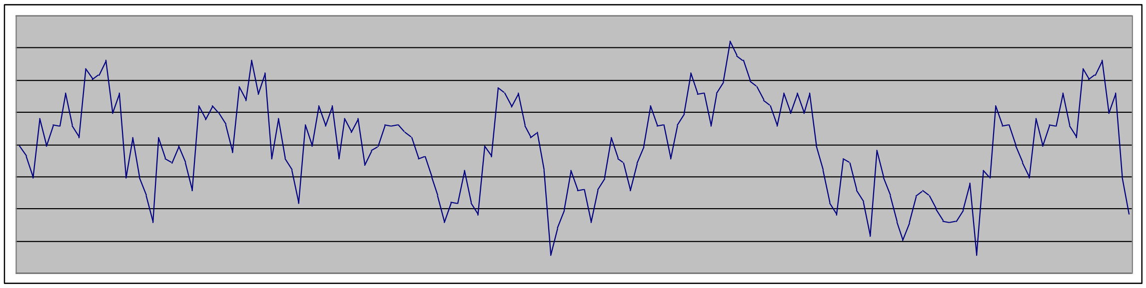
Original Graph (Upper Voice)



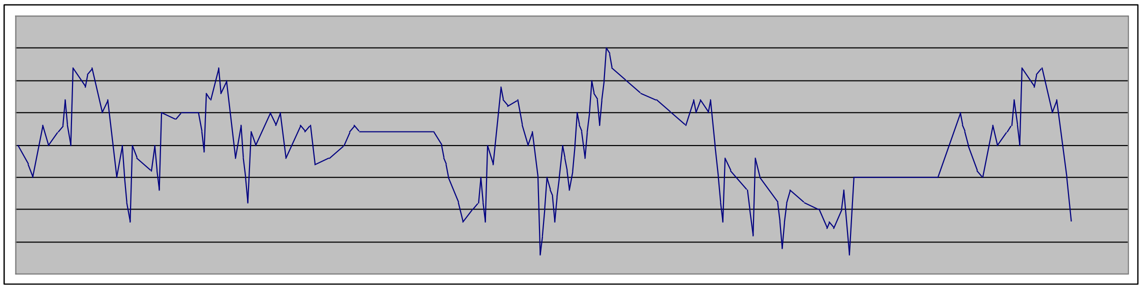
Inverse Transform (Middle Voice)



Original Graph (Middle Voice)



Inverse Transform (Lowest Voice)



Original Graph (Lowest Voice)

The attractor graph for this plot shows a somewhat erratic orbit, especially in the lowest voice and the highest end of the upper voice range. There are several strong basins of attraction spread throughout the graph.

The harmonic spectrum of the lowest voice is slightly unusual, with high amplitudes at the second and third harmonic, and with a significant amount of energy remaining through the thirty-first harmonic. After this point, the harmonics drop off and become stable until the final harmonic, which has a small spike.

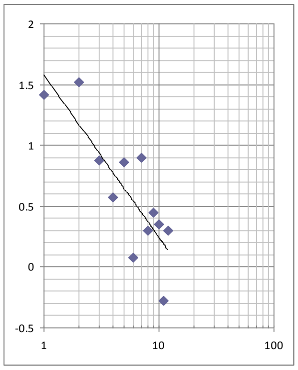
Three-Part Invention in A Major, No. 12, BWV 798 (1723)

Dimension, Madden

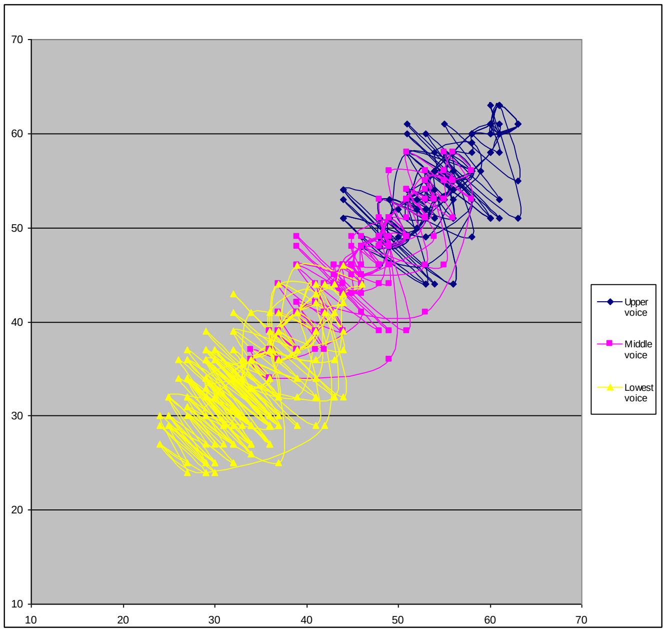
- Upper Voice: 1.1742
- Middle Voice: 1.1619
- Lower Voice: 1.2612

Dimension, Hsu

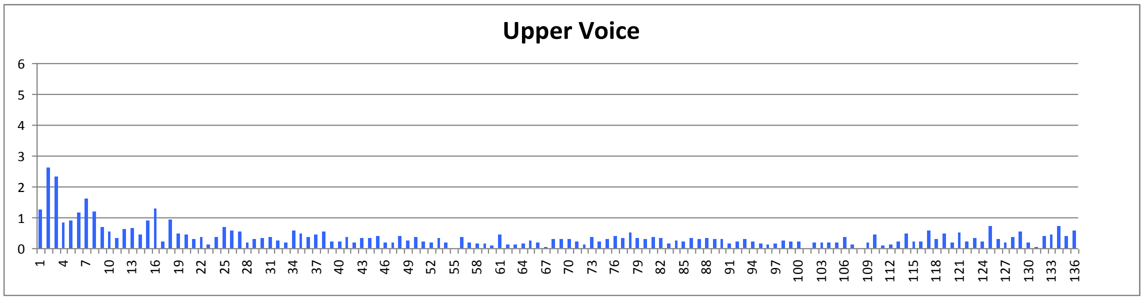
- Combined dimension: 1.3371

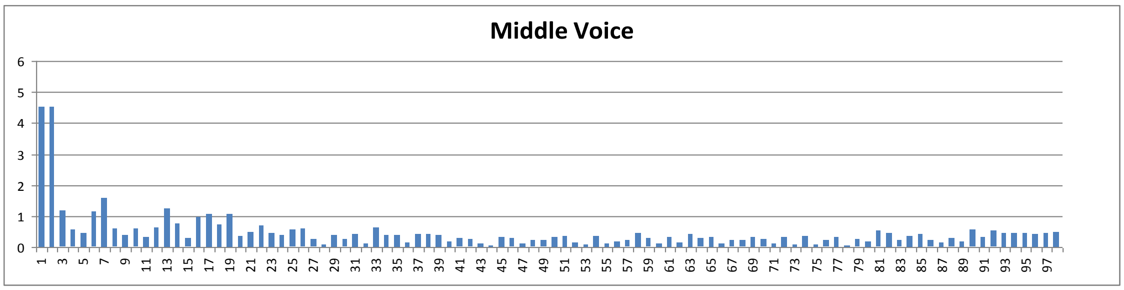


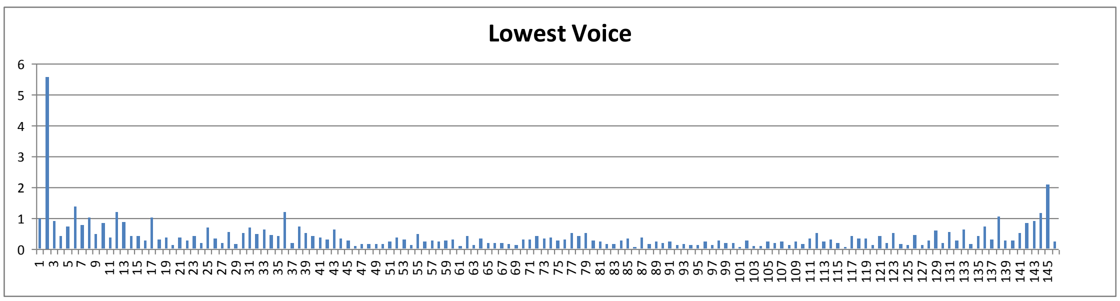
Attractor Plot



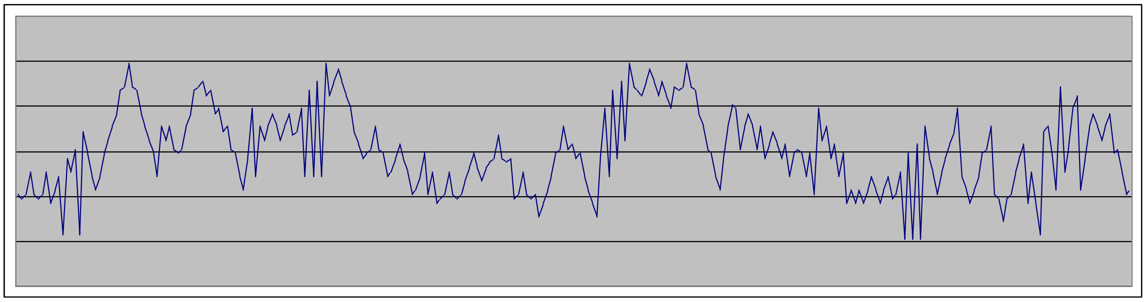
Spectral Analysis



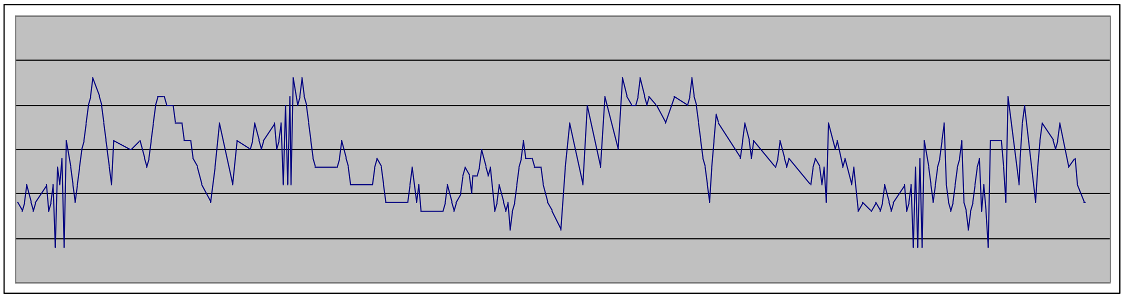




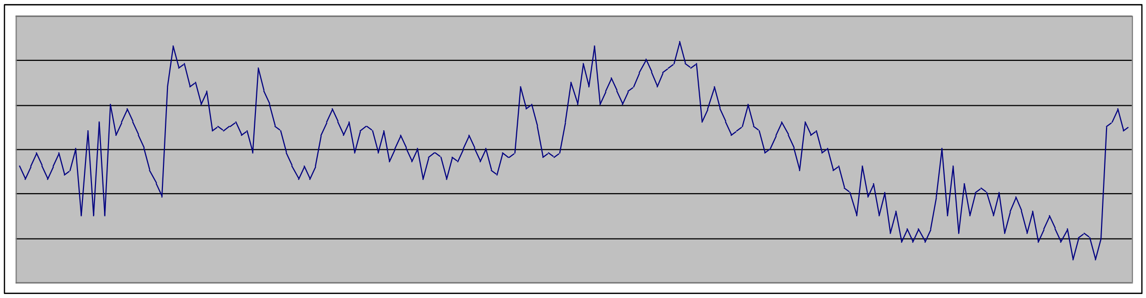
Inverse Transforms



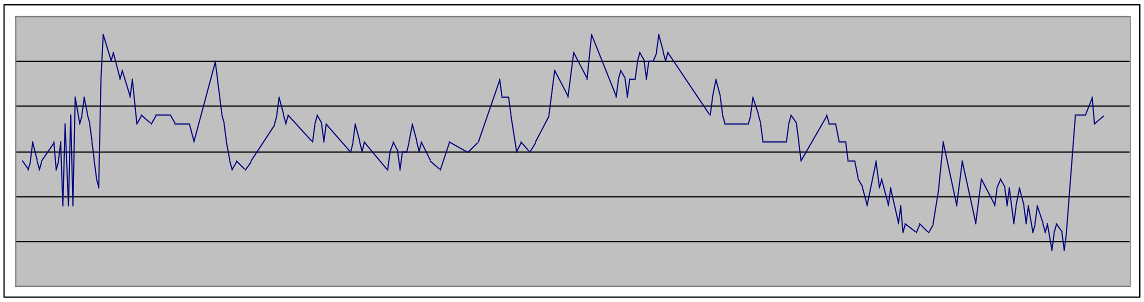
Inverse Transform (Upper Voice)



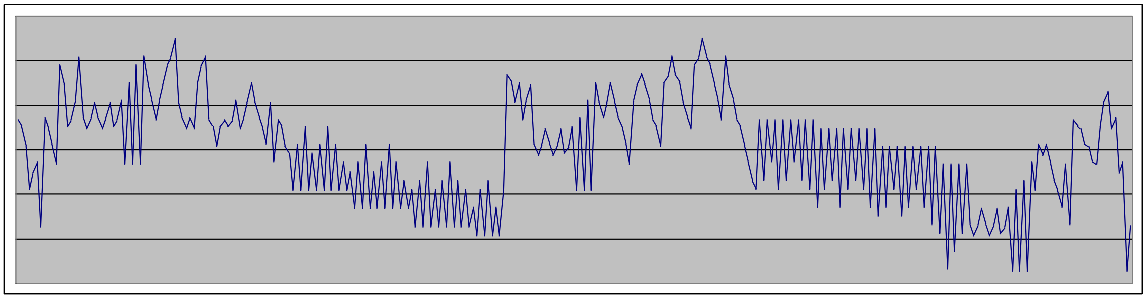
Original Graph (Upper Voice)



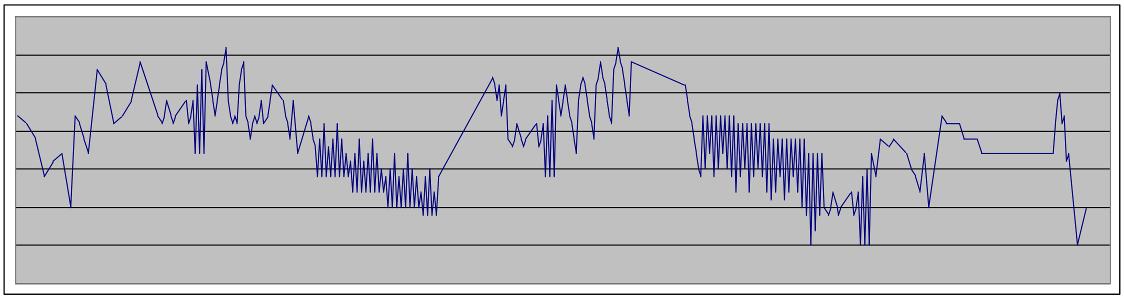
Inverse Transform (Middle Voice)



Original Graph (Middle Voice)



Inverse Transform (Lowest Voice)



Original Graph (Lowest Voice)

This three-part invention makes use of an Alberti-like figure, which creates the unusual orbital pattern. This figure is clearly used the most frequently in the lowest voice, with only a few occurrences in the middle and upper voices. However, when it does occur in the upper voices, it seems to indicate a point of attraction since the surrounding notes seem to cluster in that area. There is one more strong basin of attraction shared by the upper and middle voices that is not in the midst of the Alberti orbits.

The harmonic spectrum for the middle voice is slightly unusual in that it begins with two harmonics that are approximately the same height, differing only by 0.1451, after which the amplitudes immediately drop off and stabilize quickly with no energy in the upper harmonics.

The harmonic spectrum of the lowest voice is also rather unusual. It also begins with a high amplitude harmonic (the second harmonic), and the amplitudes immediately drop off and stabilize, but at the end of the harmonic spectrum there is a significantly interesting increase in energy.

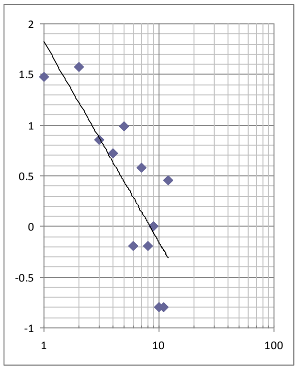
Three-Part Invention in Ab Major, No. 13, BWV 799 (1723)

Dimension, Madden

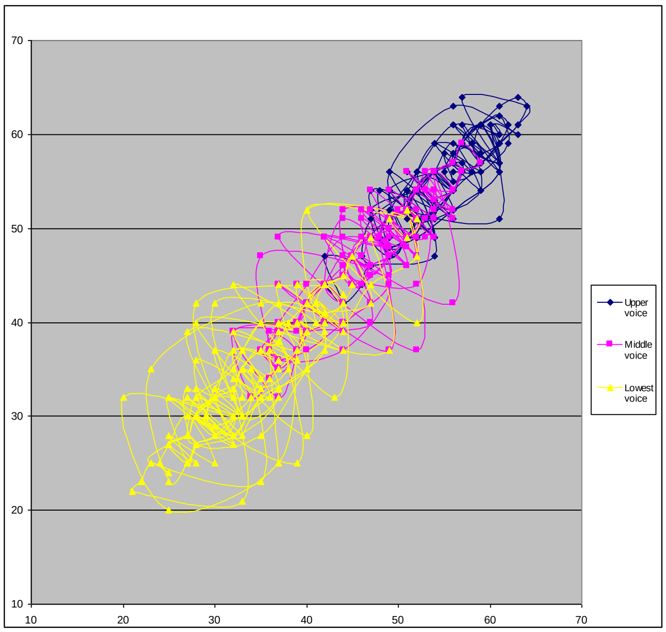
- Upper Voice: 1.1521
- Middle Voice: 1.1896
- Lower Voice: 1.2247

Dimension, Hsu

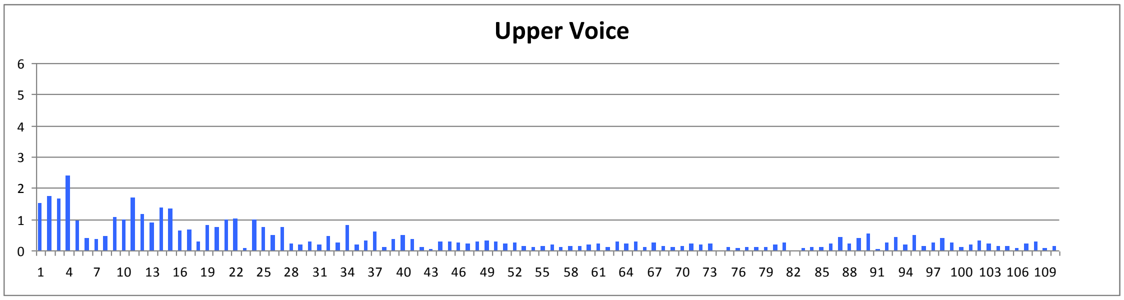
- Combined dimension: 1.9775

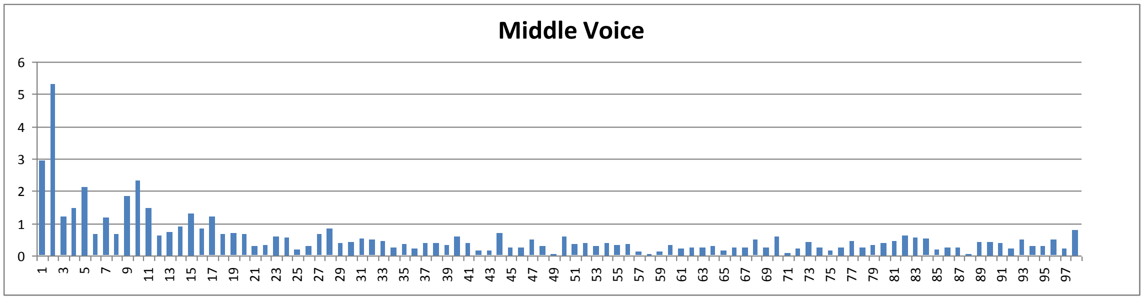


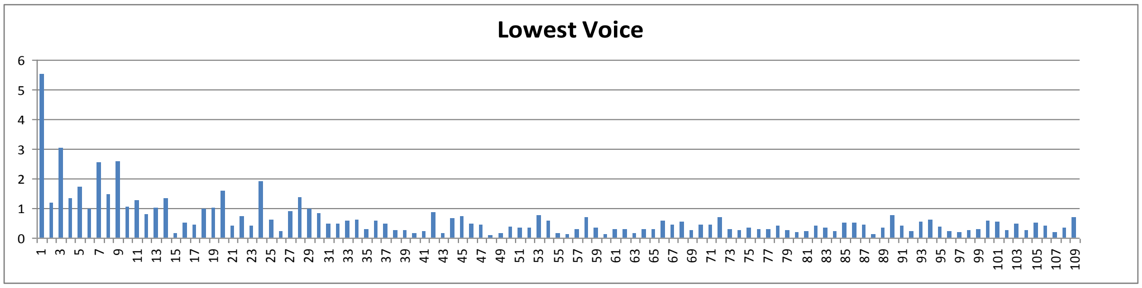
Attractor Plot



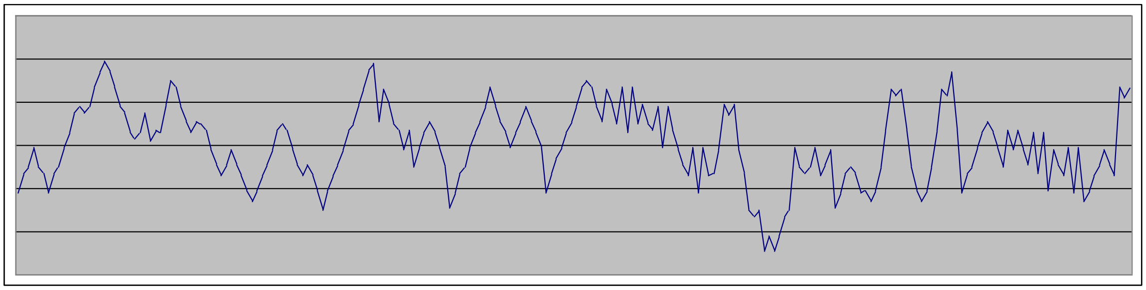
Spectral Analysis



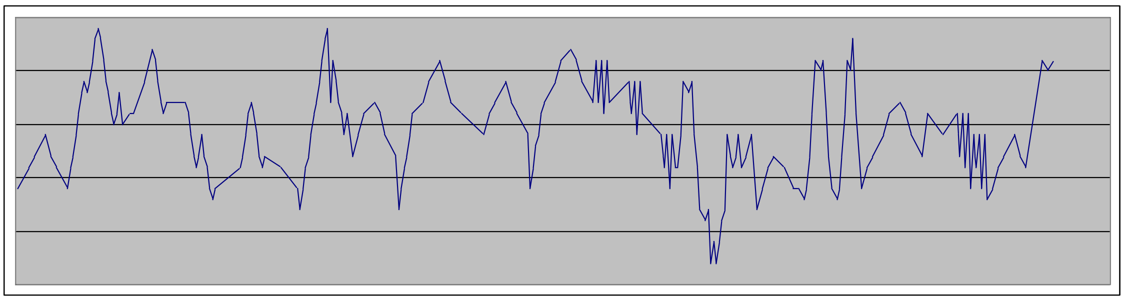




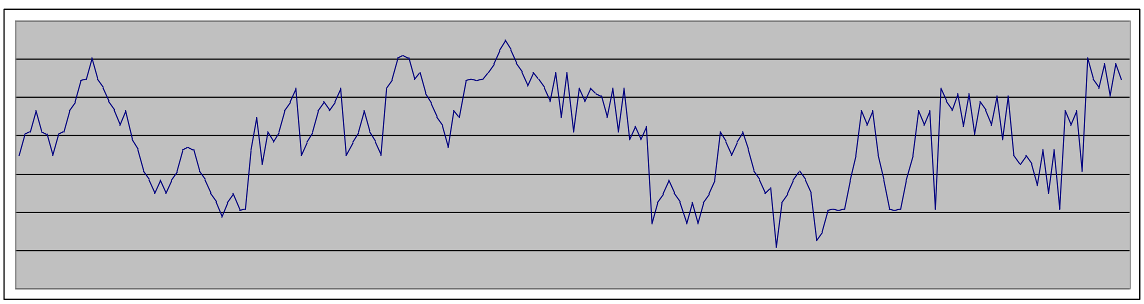
Inverse Transforms



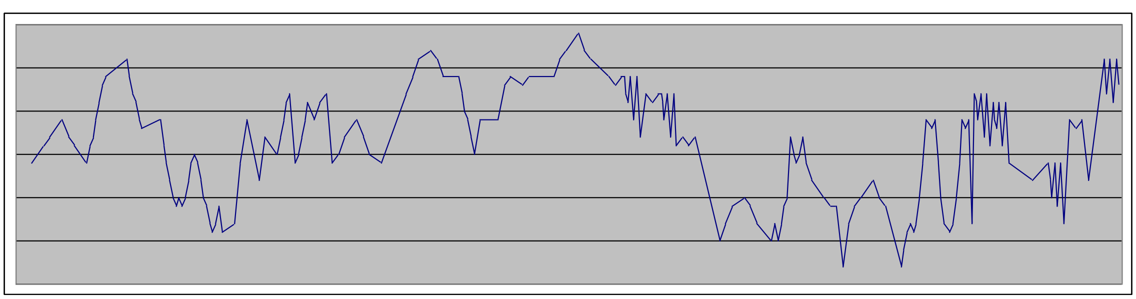
Inverse Transform (Upper Voice)



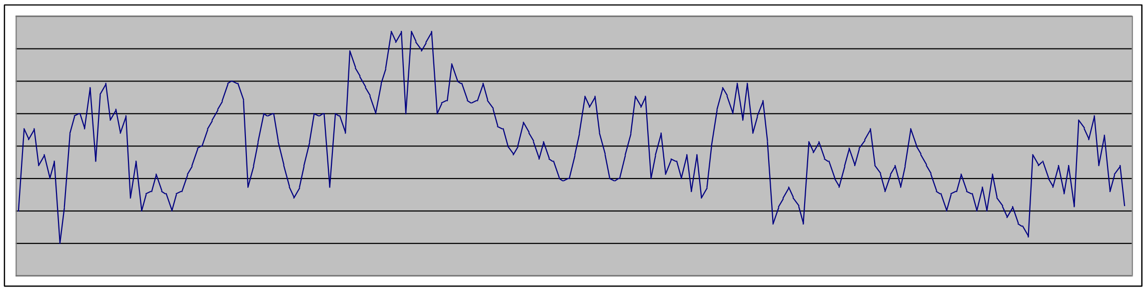
Original Graph (Upper Voice)



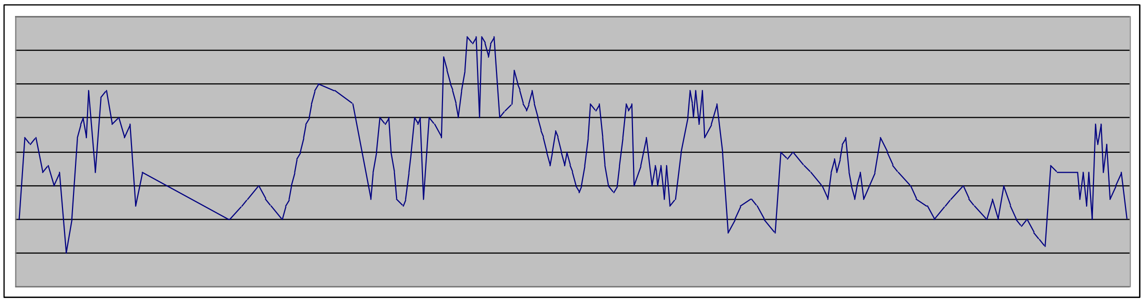
Inverse Transform (Middle Voice)



Original Graph (Middle Voice)



Inverse Transform (Lowest Voice)



Original Graph (Lowest Voice)

The motive of this three-part invention is scalar throughout, which explains the high dimension measured by the Hsu method.

The attractor graph shows an interesting overlap in the ranges of the voices, but no other unusual behavior. There are multiple basins of attraction spread throughout the system.

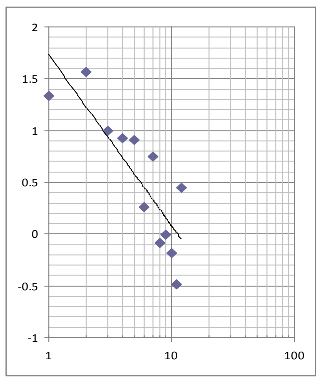
Three-Part Invention in Bb Major, No. 14, BWV 800 (1723)

Dimension, Madden

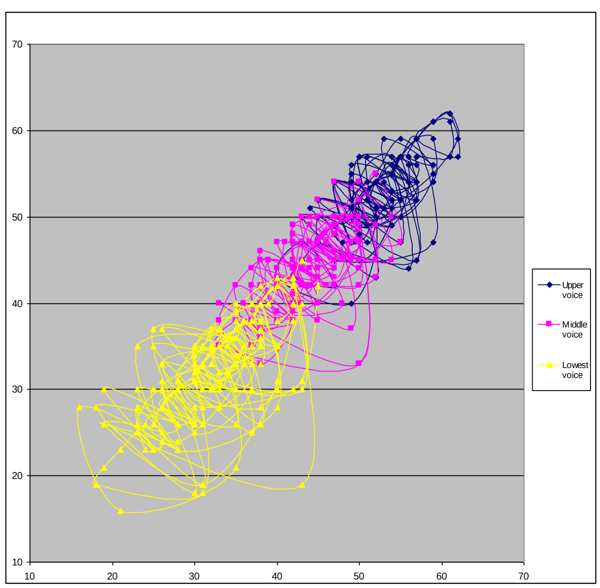
- Upper Voice: 1.2134
- Middle Voice: 1.2150
- Lower Voice: 1.2257

Dimension, Hsu

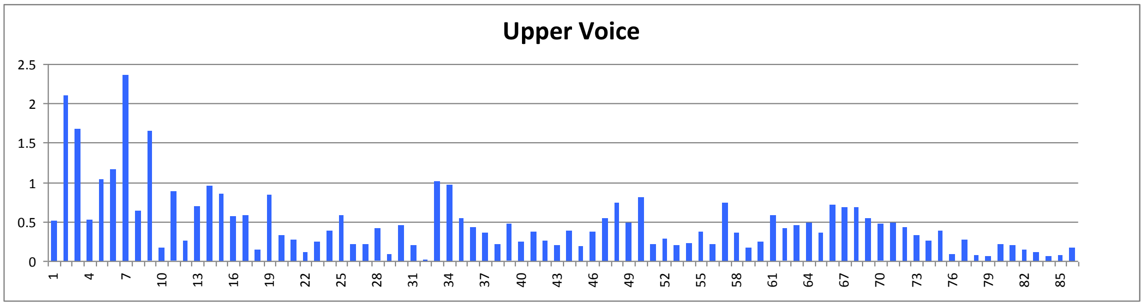
- Combined dimension: 1.6551

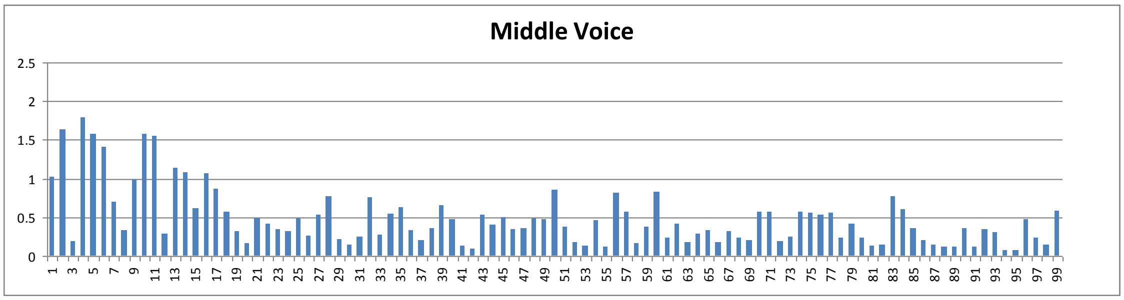


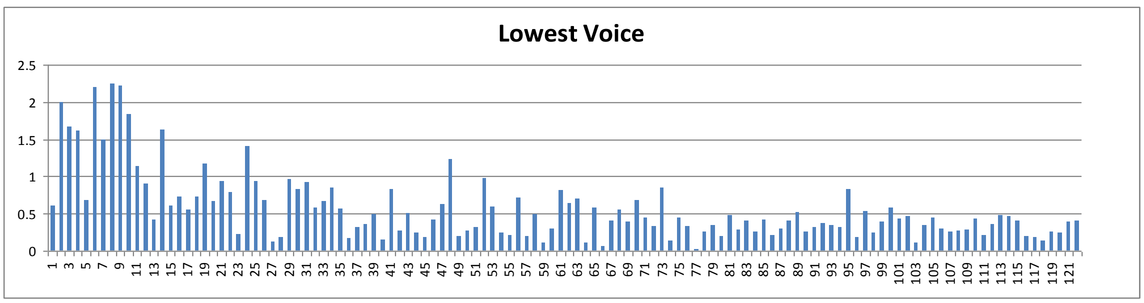
Attractor Plot



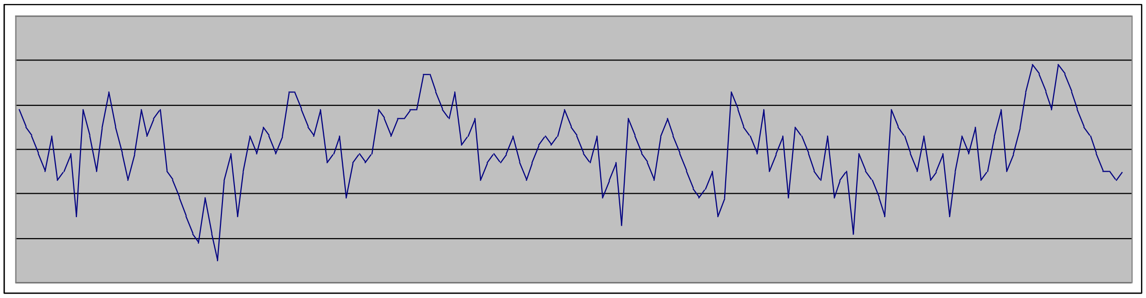
Spectral Analysis



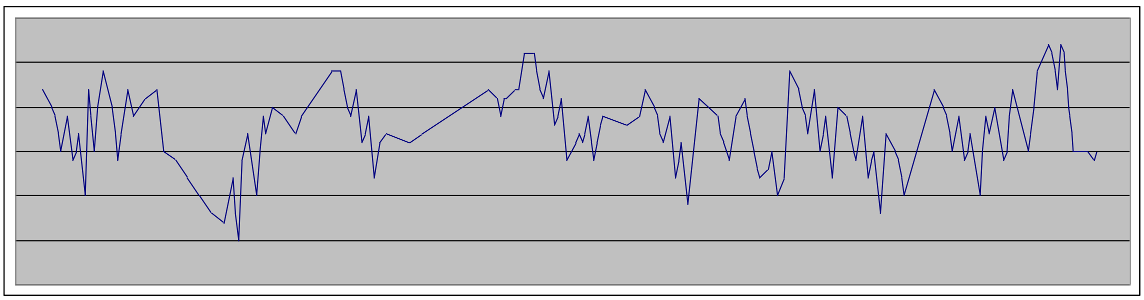




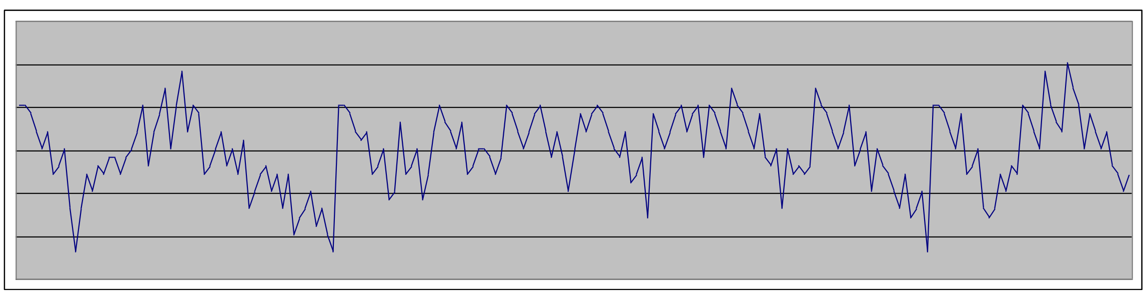
Inverse Transforms



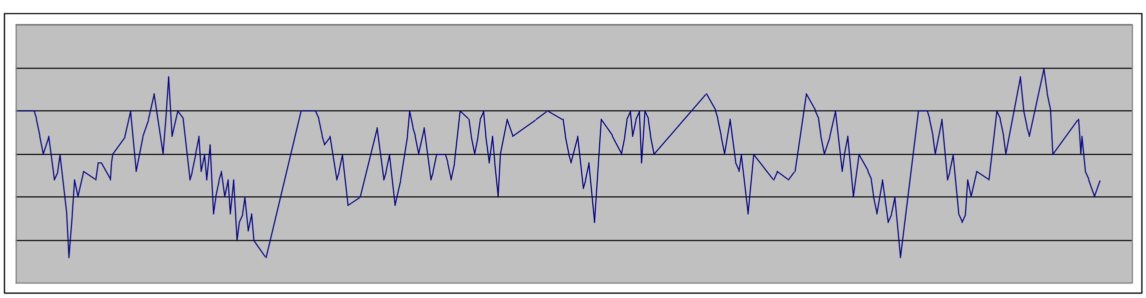
Inverse Transform (Upper Voice)



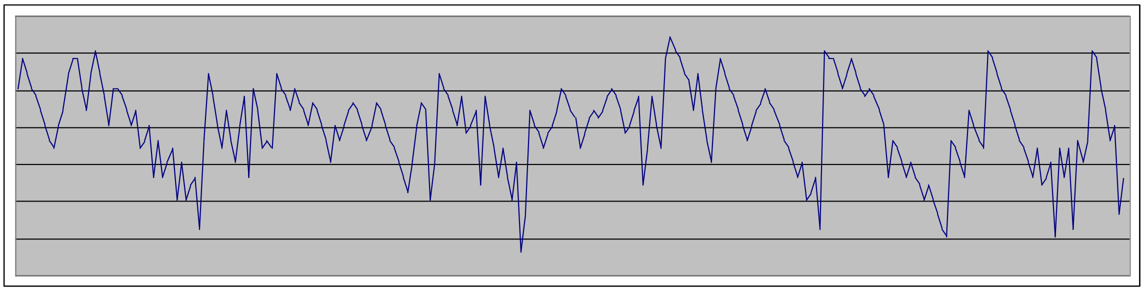
Original Graph (Upper Voice)



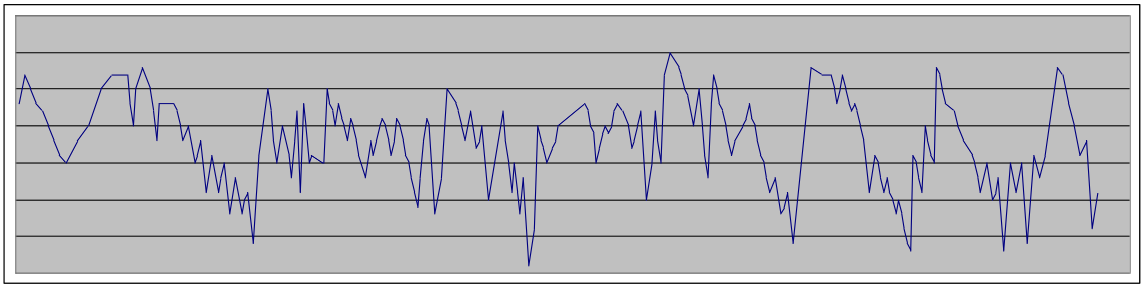
Inverse Transform (Middle Voice)



Original Graph (Middle Voice)



Inverse Transform (Lowest Voice)



Original Graph (Lowest Voice)

The high dimension measured by the Hsu method is due to the fact that while the motive for this three-part invention is not completely scalar, it has scalar sections.

The attractor graph again shows a predominance of descending large intervals, with only a few exceptions. The lowest voice is slightly more erratic than the other two voices, with more deviation from the smooth orbits, but no highly unusual behavior. There appears to be one basin of attraction within each voice, but due to the thick and relatively even clustering of notes all along the diagonal lines, these basins of attraction are not strong or striking.

All three harmonic spectrums for this three-part invention exhibit highly unusual amounts of energy throughout the upper harmonics, indicating a more complex sound curve. Noting the scale of the amplitudes, these are all relatively small amplitudes, but the amplitudes stay high throughout the spectrum rather than stabilizing. Although none of the spectrums ever truly stabilize or drop off, the distribution throughout the spectrum is not completely even, so this is not an indication of an angular or random line, just a more complex sound curve than has been seen up to this point.

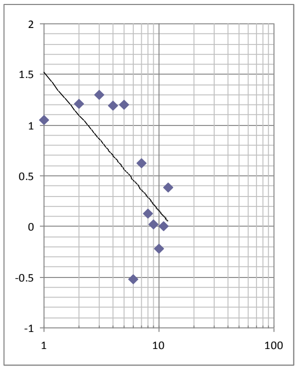
Three-Part Invention in B Minor, No. 15, BWV 801 (1723)

Dimension, Madden

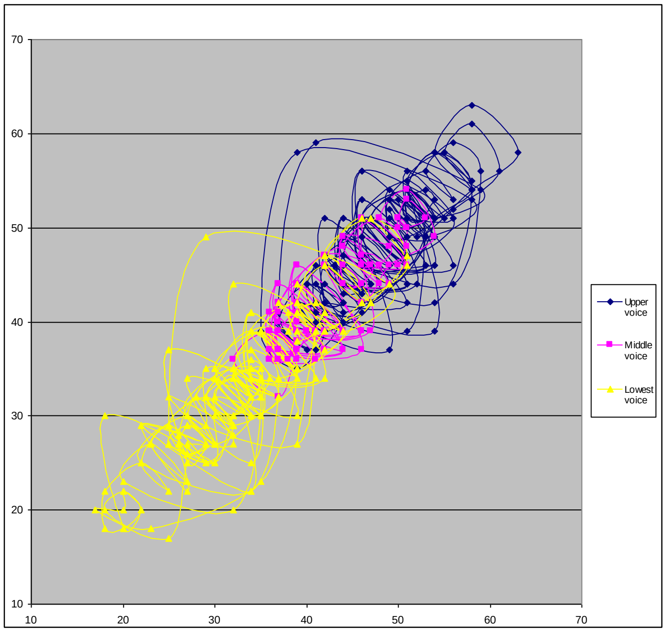
- Upper Voice: 1.2245
- Middle Voice: 1.2017
- Lower Voice: 1.2314

Dimension, Hsu

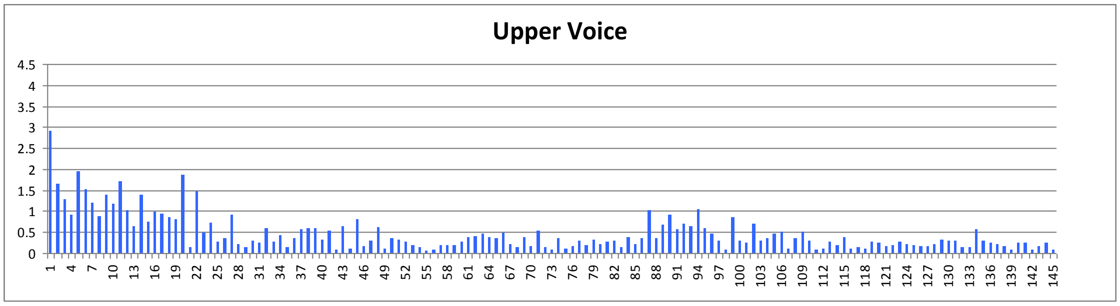
- Combined dimension: 1.3636

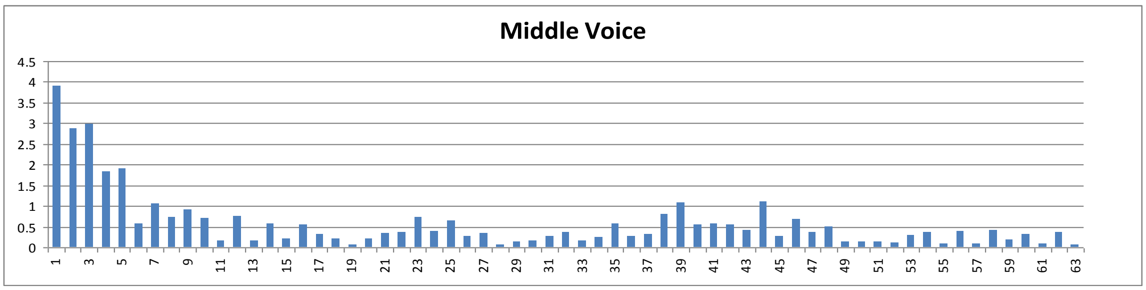


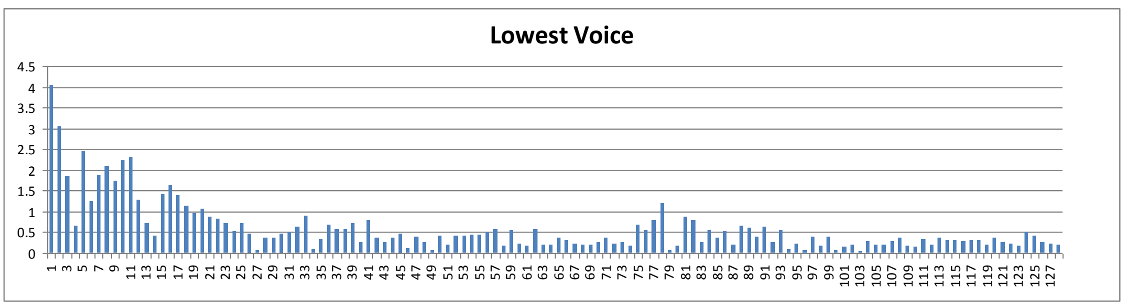
Attractor Plot



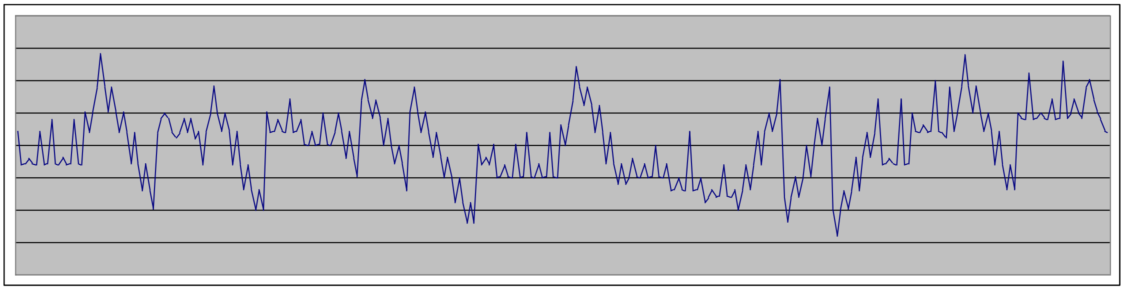
Spectral Analysis



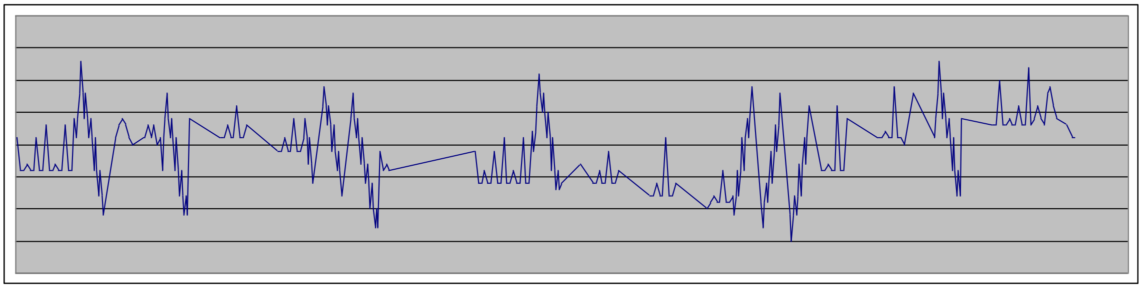




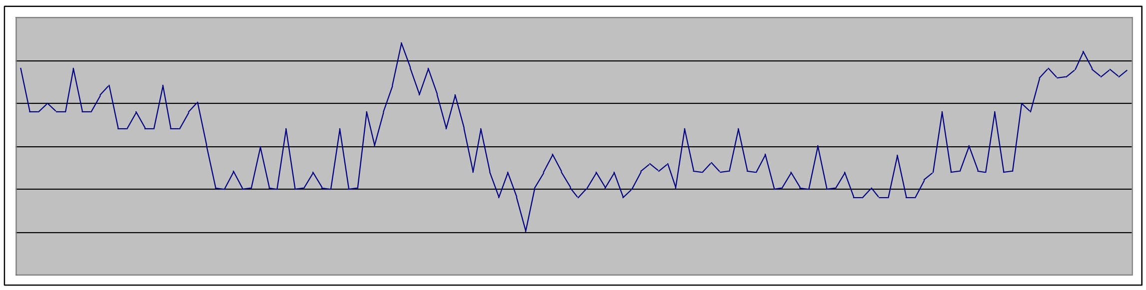
Inverse Transforms



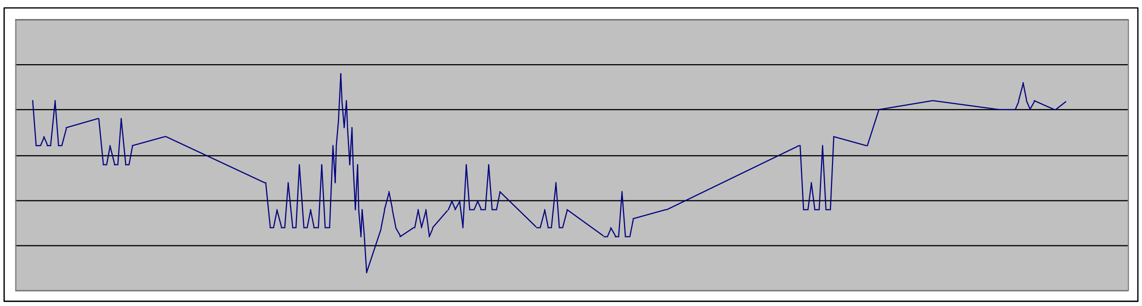
Inverse Transform (Upper Voice)



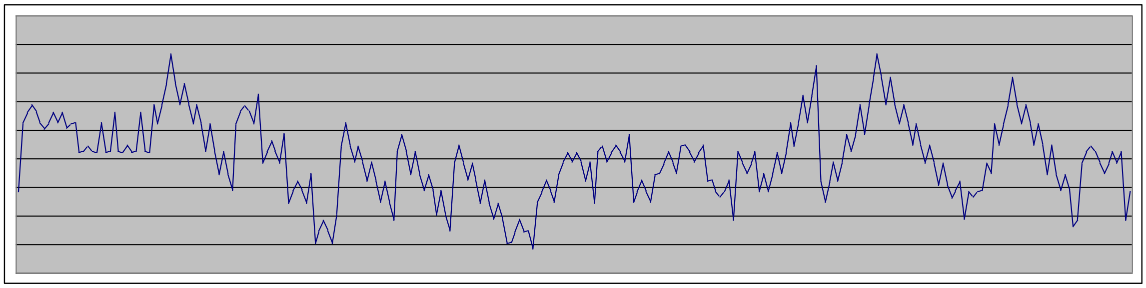
Original Graph (Upper Voice)



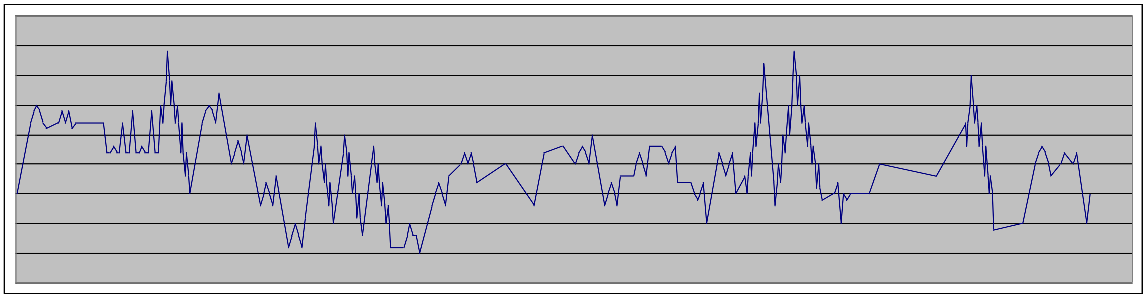
Inverse Transform (Middle Voice)



Original Graph (Middle Voice)



Inverse Transform (Lowest Voice)



Original Graph (Lowest Voice)

The attractor graph clearly shows the predominant role of the upper and lowest voices in this three-part invention. It is also interesting to note that despite the subordinate role of the middle voice, the dimension of the middle voice is not significantly different from the other dimensions. Aside from this, the attractor graph does not show any unusual behavior except for a few erratic orbits at the top and bottom of the spectrum. There are multiple strong basins of attraction spread throughout the graph.

The harmonic spectrum for the upper voice begins with a fairly normal sequence of amplitudes, although the overall amplitudes are a bit higher than normal. The amplitudes drop off and appear to stabilize after the twentieth harmonic, but there is a significant increase in amplitudes from about the eighty-sixth harmonic to the one hundredth harmonic, with no other significant amounts of energy in the rest of the upper harmonics.

The harmonic spectrum of the middle voice has a similar structure beginning with five high amplitude harmonics, after which the amplitudes stabilize until a slight but significant increase in amplitude from the thirty-eighth harmonic through the forty-fifth harmonic. Besides this slight increase there is no other activity in the upper harmonics.

The lowest voice spectrum has a similar pattern. The spectrum begins with several high amplitude harmonics, which stabilize at about the twenty-fifth harmonic. There is an increase in amplitude from the seventy-fifth through the seventy-eighth harmonic, and this increase is followed by a sudden drop at the seventy-ninth harmonic, after which the harmonics increase slightly once more and then stabilize again. There is no other significant activity in the upper harmonics.

Works Cited

Benson, David J. *Music: A Mathematical Offering*. New York: Cambridge University Press, 2007.

Chatterjee, Sangit and Mustafa R. Yilmaz. “Chaos, Fractals and Statistics.” *Statistical Science,* 7, no. 1 (1992): 49-68.

Falconer, Kenneth. *Fractal Geometry: Mathematical Foundations and Applications.* Chichester, England: John Wiley & Sons, 1990.

“Fractals and Fractal Geometry.” Thinkquest. 5 April 2008. <http://library.thinkquest.org/3493/frames/fractal.html>.

Green, Douglass M. *Form in Tonal Music: An Introduction to Analysis*. 2nd ed. New York: Holt, Rinehart and Winston, 1979.

Grout, Donald J., and Claude V. Palisca. *A History of Western Music*. 6th ed. New York: W.W. Norton & Company, 2001.

Hastings, Alan, Carole L. Hom, Stephen Ellner, Peter Turchin, H. and Charles J. Godfray. “Chaos in Ecology: Is Mother Nature a Strange Attractor?” *Annual Review of Ecology and Systematics,* 24 (1993): 1-33.

Hsu, Kenneth J. and Andreas J. Hsu. “Fractal Geometry of Music.” *Proceedings of the National Academy of Sciences of the United States of America* 87, no. 3 (1990): 938-941. JSTOR. Jarrett Library, Marshall, TX. 6 April 2008. <http://www.jstor.org/>.

Larson, Ron, Robert Hostetler, and Bruce H. Edwards. *Calculus: Early Transcendental Functions.* 4th ed. Boston: Houghton Mifflin, 2007.

Madden, Charles. *Fractals in Music: Introductory Mathematics for Musical Analysis.* 2nd ed. Salt Lake City: High Art Press, 2007.

Mandelbrot, Benoit. *The Fractal Geometry of Nature*. Rev. ed. New York: W.H. Freeman and Company, 1983.

Mathes, James. *The Analysis of Musical Form.* Upper Saddle River, NJ: Prentice Hall, 2007.

Nagle, Kent B., Edward B. Saff, and Arthur David Snider. *Fundamentals of Differential Equations*. 6th ed. Boston: Pearson, 2003.

Solomon, Larry. “The Fractal Nature of Music.” Solomon’s Music Resources. 2002. 7 Feb 2008. <http://solomonsmusic.net/fracmus.htm>.

Thomsen, Dietrick E. “Making Music—Fractally.” *Science News* 117.12 (1980): 187+190. JSTOR. Jarrett Library, Marshall, TX. 4 April 2008.

<http://www.jstor.org/>.

Weisstein, Eric W. "Fourier Series." *Math World*--A Wolfram Web Resource. <http://mathworld.wolfram.com/FourierSeries.html >

Wold, Milo, et al. *An Outline History of Western Music*. 9th ed. Boston: McGraw Hill, 1998.

Appendix
Email Correspondence with Charles Madden, author of *Fractals in Music: Introductory Mathematics for Musical Analysis*[83]

- -----Original Message-----
  From: Jennifer Shafer
  To: ChasMadden@aol.com
  Sent: Sun, 31 May 2009 5:06 pm
  Subject: Fractals in Music: Fourier Analysis
  Mr. Madden:
  I have just completed my junior year as a piano performance major at East Texas Baptist University. I have been working on an honor project this year that is based primarily on the infromation in the second edition of your book, *Fractals in Music: Introductory Mathematics for Musical Analysis*.
  The intent of my honors project is to analyze the inventions and sinfonia by Bach using several of the types of analysis discussed in your book. If you don't mind, I would like to ask you some questions about the Fourier analysis chapter. I have been trying to work through this chapter for some time, and there are some points that neither I nor my advisor (in mathematics) can understand.
  I was able to work (and, I thought, understand) the example given in the first part of the chapter, pp. 196-205. Before trying to apply this concept to the Bach pieces that I have been working on, I decided to try to analyze th e Dodge brown music, since it was a short piece that I could check my work on. This is where I ran into some problems.
  On all of the pieces that I have analyzed so far, I have used a pitch numbering system that begins at the bottom of the keyboard (making middle C have pitch number 40). I noticed in chapter 6 that while the book uses numbers to designate the pitches, the numbers for each pitch are not always consistent from one piece to the next, implying that there are different systems used to determine the numbers? Is there a correct way to number the pitches? I found that I came much closer to matching the An, Bn, and Cn numbers on the Dodge brown piece when I used the numbers given on page 131, which are consistently 40 less than the numbers I originally used.
  In figure 9.12 (and in several subsequent figures), the "x"-axis is labeled as "pitch/frequency". Does this mean that these points are the pitch numbers divided by the frequency numbers? Initially I used the pitch numbers given on page 131 for this music because in the last paragraph of page 204, the book stated that "the samples are the pitches themselves." Upon noticing the labeling of this axis, however, I went back and used a pitch numbering system beginning at the bottom of the keyboard (thus giving the pitches 49, 48, 49, etc. for this piece), and divided the pitches by the frequencies. I haven't been able to make this graph look like the one shown in the book, although a graph with either the pitch or the frequency on the "x"-axis looks the same as the one given in figure 9.12. I am not sure, then, which numbers I should use as my f(t) numbers in my transform.
  I have also not been able to figure out how to determine the S and T values for the Dodge brown music. I have tried to use T=25, since that is the number of samples in the Dodge brown piece, but I have not been able to make my numbers co me out identically to the ones given in the book.
  I hope the above questions make sense. I would greatly appreciate any explanation you can give. Thank you.
  Sincerely,
  Jennifer Shafer

- Dear Jennifer,

  Thank you so much for your note. I am glad that you are working with this material and really appreciate this feedback. I will need perhaps a day to scope out your question and its implications and will get back to you.

  Charles
- Dear Jennifer,

  You're to be commended. Chapter nine is dense. It must require considerable concentration to understand it at all. You do well to test your understanding by trying to duplicate the Dodge brown music. The lesson there is to always check. We've all been known to err! (I recently tried to read a seemingly very important and highly-regarded book that had the entire premise backward. I'll spare you the details. Thus none of it can be true, but it took a while to figure out what was wrong.)

  Pitch numbering is rather arbitrary. It has not been standardized completely, but in advanced analysis we usually start at zero because it facilitates twelve-tone analysis. It's in essence a "moveable do" system. It might be good to always start at zero for the key-note, but then you get negative note numbers. This is not fatal, but it could be inconvenient. Or the lowest note. Starting every piece at A0=0, as in Scientific Pitch Notation (see p. xix) would sometimes make pitch values unnecessarily large, but notice that I had to do that with the Xenakis piece on p. 128. Also, you will out find sooner or later that pitch numbering systems for pianos and synthesizers usually start at A0=0, not 1. Whichever you start at, the results should be comparable, since it is a linear system (it is additive) and there won't be any distortions due to range. Thus, it's mostly a matter of convenience. The one thing that is required is 1 2 numbers for each octave (usually 0-11, 12-23, etc.) in order to pick up all possible chromatic pitches.

  The numbering for the Dodge music began at C4=0 (middle C). How do I know that? It's buried on p. 131! That's a revision that needs to be made in the next edition: Make it easier to follow! Along with an explanation about the numbering. (I thought I did that, but I can't find it now. Maybe I imagined it.) G4=0 would also work.

  The slashes on pp. 205 and 207 in this case are intended to mean "or," not division. That is an overloaded symbol that I had not noticed, like parentheses for functions or grouping, such as $f(x)$ ("ef of ex") or $x(x)$ ("ex times ex"). Mathematics is filled with overloads like this and

[83] Typographical errors have been left unchanged from original e-mails.

it takes a while to learn to tell the difference. You did right to try the Dodge music with the pitches, which in this instance I had considered to be the same as frequency. However, it should just be "pitch." "Frequency" is wrong and would not give a correct result.

Summations (sigma) are from 0 to T-1. See p. 201. You start at time zero, and after one unit of time, t=1. In the Dodge "music," there are 25 notes, but (T-1)=24. Thus, there are 24 units of time and S and T are 24. Try this number and any pitch numbering scheme you like, and if you get a different result for the *Cn*'s (more than a decimal point or two in the fourth decimal place), let me know. Also, make sure you understand the summation formulas on p. 201. It might be wise to review them with your advisor.

Notice that there is an analysis of the spectrum of the upper line of *Invention No. 1*. If you don't get this result, let's talk.

By the way, this looks like a huge project. Are you sure it's not a master's thesis?

Let me know how it goes.

Regards,

Charles

- It's hard to see how I could have made the discussion any more difficult. You have revealed the need for more revisions. Thank you.

  I said: "In the Dodge music, there are 25 notes, but (T-1)=24. Thus, there are 24 units of time and S and T are 24." Well, duh, T is not equal to T-1. It should be T=2S=50. The book is wrong, too. T=2S.

  This is explained clumsily for the triple-sine example starting at page 200, where the highest frequency is 5 units, times 2 samples per unit, or S=10, and T=2S=20. Now, in the Brown music, we have S=25 samples, so T=50. That is what I used in the spreadsheets.

  I apologize for the error. Please let me know if anything else is unclear.

  Charles
- -----Original Message-----
  From: Jennifer Shafer
  To: chasmadden@aol.com
  Sent: Mon, Jun 29, 2009 9:38 pm
  Subject: Re: Correction to discussion of Brown music
  Mr. Madden,
  I wanted to let you know that I think I have finally worked out all of the bugs in my working of the Dodge brown music. I am finally getting the same answers as are given in the book. I have not yet applied the formulas to my analysis of the inventions, but I am hopeful that I have finally fixed all the errors in my formulas.
  One more question: As a recommendation, how many coefficients/terms should I work out? I know that the more times I iterate the basic formulas, the more accurate my final equation will be, but is there a standard or minimum number of terms that I should start with?
  Thank you again for taking the time to help me with this project. I greatly appreciate it.
  Jennifer Shafer

- Dear Jennifer,

  Good for you, and thanks for your questions. You are the first to do so, and it gives me a chance to find out where things could be clearer. In fact, the question about the number of coefficients made me go back to the far reaches of my memory (that cave gets darker the older I get). In fact, there are several unstated assumptions here that even I could not remember at first.

  One assumption that is essential to know can be gathered from figure 9.12 for the Dodge brown music: There is one low-frequency sinusoid that dominates the motion. Its amplitude is given by C1. Its amplitude is shown on the far left in figure 9.13. It would correspond to the left half of the highest sine wave in figure 9.6.

  Another assumption is that there are several other sinusoids at frequencies of integer multiples of the lowest frequency, perhaps an infinite number of them, but our method causes them to repeat periodically and in reverse after half the number of samples. Their amplitudes for the Dodge brown music are given by C2 to C12 in table 9.5. These higher frequencies literally add together to make up the reconstituted waveform in figure 9.14. They would be comparable to the two faster tones in figure 9.6, which add up to the wavy form shown in figure 9.8. You can see a direct relationship to the height of the bars in figure 9.13 and the C-values in table 9.5.

  So, the direct answer to your question is "As many as you want," but it is not informative to use either more or less than one-half the number of samples, in this case 12 or 13 (13 would be the first repeated value). As you can see in some of the other examples in this chapter, there may be high-frequency harmonics that make a difference. A general rule for interpreting these things is that high-amplitude low-frequency harmonics give a long slow sweep to the melody while high-amplitude high-frequency harmonics give a more jagged contour and/or add to the curvilinear appearance. You might test this by sketching the lowest-frequency waveform and then the second-lowest right over it, with the proper proportions, and add their values to get a third waveform resultant. It should look a tiny bit like the original, although a lot smoother.

  Also notice that we are talking about melodic contours here. The Inventions might have to be done using one analysis per voice. I haven't tried using this for multiple voices because I did not think that we could learn anything from that, although I'm not sure anymore what I did in figure 9.32 for the first Invention. If you need to know, I can dig into my files.

  Let me know how this turns out. Best wishes to your advisor. Does he understand any of this?

Charles

- Incidentally, we are not iterating, we are calculating individual values that are then added up. Iteration refers to using previous values to successively approach an assymptote, such as a limit.

  Charles

- -----Original Message-----
  From: Jennifer Shafer
  To: chasmadden@aol.com
  Sent: Mon, Aug 31, 2009 4:19 pm
  Subject: Fourier Analysis
  Mr. Madden,
  I have a few more questions for you, if you don't mind. One more on the Dodge example, and then on the inventions.
  Once I had set up a spreadsheet with the formulas to perform the inverse transfrom, I found that I had to take the t values for the equation in table 9.4 out to 48, instead of the original 24. Did I make a mistake in my formulas? I cannot figure out why I would seem to need twice as much "time" in the inverse transform.
  Also from table 9.4, what does the f(t)=2.26 mean? I thought it had something to do with margin of error, but I am not sure how to determine it.
  I have also begun trying to apply the process to the inventions. I am wondering about the T and S values for the inventions. For instance, in Invention No. 1 (right hand), I have 237 intervals (therefore, 238 samples), so is this the correct number to use for S, thus making T=476? If this were the case, then the numbers s hould start repeating at n=120, right? (Mine don't.)
  On a slightly different note, in each of the inventions I measured time in the smallest (or sometimes next to the smallest) note value--typically 16ths. This allowed me to build my spreadsheets with my note/time graphs easily, but I am wondering if that doesn't work for this application because I often have one note lasting for more than one 16th notes. This makes my time column skip numbers (i.e. 1, 3, 4, if the rhythm was eighth-eighth-sixteenth). Do I need to rework my spreadsheet to include each individual time value (1, 2, 3 . . .), and just "repeat" the note for its duration?
  I am not sure if this makes sense as I reread it, so I am attaching the file for Invention No. 1, hoping that that may clarify things. Thank you for the time you have put into answering my questions.
  Jennifer Shafer

- This is proving to be a huge project, isn't it? I hope you are finding it worthwhile.

  I'll study it out and get back to you soon.

  Regards

  Charles

- Jennifer,

  Thanks for the spreadsheet. You've done some good work there. It looks like you understand the Hsu graph. I did not. I'd like to know how he makes it run to the left like that. It's apparently an engineering approach that I have seen only a very few times. If you can show me how it works, I would be grateful.

  I don't know why you would have to go out to 48 on the Dodge. Can you send the spreadsheet?

  In table 9.4, 2.26 is A0. It's a "DC Bias," as they say in electrical engineering, that is, an ofset from the *x*-axis. See pages 196-7. It's only in the equation for drawing the waveform and I didn't spend much time explaining it because we want the Fourier Transforms instead of the reconstructions. I stopped showing the reconstructions after figure 9.32. If it's a fault to ignore it, let me know and I'll try to do something with it in the next revision.

  I would rather count pitches than intervals. Hsu's contention that what counts is intervals is simply one way of working, i.e., his opinion. Intervals will not give us any information in the Fourier work; it's all about frequencies/pitches. I feel strongly that Hsu missed the boat in parts of his analyses.

  I don't recommend counting sixteenth notes. That would add a dimension (time) that would greatly complicate the project and probably change the results. It would require multi-variate analysis, which I don't expect to be able to tackle in this lifetime. The objective in all this was to isolate the pitches from all other complications to provide a one-dimensional analysis. I know that some analysts insist that it's not an analysis without the durations. I disagree. You can learn something from one dimensional analysis; it's the scientific way. If they want additional dimensions, then this is not the right system to use.

  As for time as we use it, it is one unit of time per pitch, solely to project the pitches onto a plane. Throughout the book, pitches are given as "raw" (as my notation program calls them) to divorce them from duration. If a pitch is repeated in the score, it is repeated in the analysis, but not if it is held. The "sample values" are the notes themselves: same number of samples as notes.

I'm not sure how I gave the impression that your S and T should be so large. There should be 118 harmonics (half the number of pitches). I don't know what 175 across the top is, or 476 down the side. I also haven't yet studied your use of the sines and cosines, and whether you should have a squared sum at the bottom. So I need another day. I have attached my spreadsheet.

Regards,

Charles

- You have processed only the cosines. You also need a spread for the sines. You need to also have a square root of the sum of squares of the sines and cosines. This gives the coefficients for the Fourier display. See the spreadsheet I sent last (2MB).

- A couple of thoughts.

  Dodge should start repeating <u>in reverse</u> after 25 harmonics. So maybe your effort was correct, except that it's pitches not intervals.

  The number of columns should be the harmonic numbers that we want, which should be half of the number of pitches (again, because they start repeating in reverse). The number of rows should be the number of pitches. Also, don't forget the sine section to get the B coefficients, and to add the squares of the Bs to the squares of the As and take the square roots to get the Cs. The spreadsheet that I sent should be clear (I hope).

  I needed you three years ago before I revised this book to show me how impenetrable it is. Sheesh! Even I can't read it. I have to look at the spreadsheets! Which, of course, aren't available to readers. This is a drawback of not being able to test it on students before foisting it off on the public.

  It's good doing business with you.

  Charles

- -----Original Message-----
  From: Jennifer Shafer
  To: chasmadden@aol.com
  Sent: Wed, Sep 2, 2009 1:36 pm
  Subject: Re: Fourier analysis
  Mr. Madden,
  Attached is my spreadsheet for the Dodge music. My charts for the the An and Bns are in the Coefficients sheet, followed in the other sheets by the inverse transform formulas, and the inverse transform graphs. In the inverse transform graph, when I only took the time out to 24, I didn't get the complete graph. However, as you said, the numbers do repeat backwards if I go to the 13th harmonic.
  My apologies--I didn't communicate clearly about my invention file. I was aware that I hadn't finished the work for the Fourier analysis on this file. I could tell that something was wrong with my numbers, so I didn't work out the sine part or the squares. The reason I took it out so far to the right was because it never began repeating, and I thought I must have made an error somewhere along the line.
  Thanks for your explanation about the time aspect of it. That makes much more sense, and I think I should be able to work that part of it out now.
  I think I am still somewhat confused about the numbers T and S. As you will see in my file for the Dodge music, I have S=25 (since there were 25 notes in the Dodge example) and T=50 (since T=2*S). Those numbers were from our previous correspondence. I was trying to take that concept and apply it to this piece. Since I had 238 pitches, then S=238, and T=476. Am I still making an error here?
  I am not sure that I completely understand the Hsu graph (or his logic to get there), and I get different numbers than he does, but for now I have been including his method with yours for comparison, and because his uses a much different method to determine the dimension. I'm not sure what you mean by "run to the left", unless you mean the logarithmic scale on the bottom?
  Again, thank you for the time you have put into answering all of my questions. You have been a wonderful help on this project.
  Jennifer Shafer

- And you have been a wonderful correspondent.

  By run to the left on Hsu, I mean that the line slopes to the left. Richard Voss's charts do that too. I have never been able to find an example in a text, and Voss won't talk to me, so I don't know what they do. I, too, get different numbers for Hsu's dimension. In discussing dimension, he mentions an empirical constant of 2.5 or so, for which I have no clue. Hsu was, however, wonderfully helpful with an example of self-similarity in Invention No. 5.

  I'll have to look at Dodge later. I've been up all night and day. My wife, too. She's a night person, but had to go to jury duty today!

  Don't worry about checking for repeating harmonics in the worksheets, unless it's helpful to you. You can be sure that's what they do. I think it's Gareth Loy's *Musimathics* that explains why that is so. Incidentally, I heartilly recommend that book, even though it doesn't relate strictly to this one. His explanations of many things musimathical are superb. Anyway, the pitches represent one(-half) cycle of the lowest pitch (so it's the fundamental harmonic) and the front and back end-points create a sharp-edged window, which causes a distortion. That's the backwards-repeating bit.

  I confess that I have confused myself, too, about T and S. That is one thing that needs revision. It makes no sense! (Ouch, did I say that?) But notice their use in the formulas of the spreadsheet I sent for the Invention. What I meant is very clearly applied there. Now I have to figure out how to say it clearly. I think that the confusion is coming from trying to show how to sample a waveform and getting the Nyquist cut-off rule tangled up with it. I will probably cut that part out next time. It's not wrong, it's just garbled. (I had been dissed by a reviewer of the first edition and was over-sensitive about it.) So, the rule for figuring out T and S is to maybe forget about it and to realize that we're looking for half the number of pitches for the top row (the harmonics--I got 118, you have 119, I guess--small difference in counting) and all the pitches for one of the left columns (we have used different columns)--238 total pitches--like I explained in an earlier email, and using the spreadsheet as a guide. I think 476 is an error. You'll see what T is by looking at the code in the very rightmost upper cell and what the other number is by looking at the divisor in one of the very lowest and leftmost cells.

I have to say that I should have provided more than one spreadsheet in the book than just the Dodge. Big mistake.

I hope that's a bit more clear. Keep me posted.

Charles

- -----Original Message-----
  From: Jennifer Shafer
  To: chasmadden@aol.com
  Sent: Thu, Sep 17, 2009 1:55 pm
  Subject: Fourier analysis
  Mr. Madden,

  I apologize if you get this message twice. I sent it about a week ago, but I am not sure that it went through. I realize that you are probably busy and may not have had time to respond, but I decided to resend it just in case it didn't go through the first time.
  Thank you again for the time that you have taken to assist me with
  this project. I would never have gotten this far without your help.

  I think I have found my mistake in the Dodge analysis after reading
  your last e-mail. I used 50 in one place in the formula I used to
  get the coefficients where I should have used 25; therefore, my time
  component in the end was also off by a power of 2.

  I finally got my file to work for the Bach Invention No. 1, but once I finished, I found that I have yet another dilemma. I am working on writing my paper detailing the process for this project (as well as discussing the mathematical id eas behind the different types of analysis), and I find that I have essentially no idea what the spectrum graph is telling me. I have read and re-read through all of chapter 9, and I can't figure out what I should be looking for in these graphs--what is significant about the results in the spectrum graph. I'm sorry--I know I am probably just really dense, but I have really come up against a brick wall on this one. Could you possibly explain this to me? I would greatly appreciate it if you could enlighten me on this.

  Also, are there any books or other sources that you could recommend
  for an elementary understanding of attractors (strange and
  otherwise)? I read/skimmed through many of the books in your
  bibliography last spring, but did not find one that I was able to
  really understand in-depth. I need a source for my paper that will
  explain attractors (just in general, not necessarily as related to
  music) on a fairly simpl e level, since that is something that I am not
  sure I fully understand, and it will be unfamiliar to most or all of
  my readers. I have so far been unsuccessful in finding a source that
  is at a sufficiently elementary level. Anything you could suggest
  would be greatly appreciated.

  Thank you again.

  Jennifer Shafer

- You are smart and the day will come, at the rate you are going, that you will encounter problems that no one can answer because no one knows. So I offer a possible method for future reference to help you get through those times. This method is slow and probably not applicable to you now, with deadlines and tight schedules, but it's what many researchers do when dealing with baffling deep problems. 1) Meditate on it. Spend days just thinking about it if you have to. Look at all the angles and try to find alternative ways to explain it. Go back over the texts and meditate again. After a while, a light dawns. 2) Be aware that many authors, including myself, do not write clearly. And they all make mistakes. There probably is not one book that is error-free. When baffled, meditate on it. That should help.

  Now, I repeat that I am happy to help. So here goes.

  I am of the opinion that my book is the clearest about attractors. Chapter One and Chapter Three have considerable material. Aristotle's problem of zeroing in on pi is an excellent example that should impress your advisors. (pp. 39ff). The attractor is the circle itself, and also the number 3.14159 . . . . Another example referred in the book is the railroad tracks approaching a point in the distance. The point is the attractor. You can see this in a line of telephone poles down the street, or the highway paint stripes ahead. The spiral on the cover of the book leads into the center. This point is an attractor for the spiral. Limits and convergence, which are referred to several times, are attractors. Attractors may be points, lines, circles, or anything toward which something tends. Tonic triads after dominant; *pp* after *decresc.*, keytones, etc.

The best book that is not super technical, might be *Fractals* by Hans Lauwerier, although it is not just about attractors. (There doesn't seem to be one that is just about attractors.) Another good one, much more daunting, but the best overall, is *Chaos and Fractals* by Heinz-Otto Peitgen et al. Don't bother with Mandelbrot. He is unintelligible.

As for why we do Fourier transforms, that's a bit tougher to explain. There are some tiny clues tucked into little nooks in the text: "Fourier coefficients can be used to describe a melody's shape in terms of its constituent sinusoids." [p. 191] "What this really measures is the amplitudes of the sine and cosine waves that make up the complicated waveform that is the melodic shape." [p. 204] "The fundamental frequency provides the slow rising and falling shape, while the smaller, higher harmonics account for the smaller movements." [p. 206] "We can hear this piece [Invention I] as undulations of five main waves supplemented by smaller waves of higher frequencies." [p. 213] "As melodies become more complicated, their spectra broaden." [p. 219] (Thus, the spectra are a measure of how rough or smooth melodies are.) "Higher amplitudes of the upper harmonics indicate greater twisting and turning in the melodies." [p. 220] "The Fourier transforms discussed here provide in sight into the effect of high- and low-frequency components on melodic shapes." [p.220]

For a good example of the effect of the harmonics, see figure 9.36, where Chopin's piece is 39 repetitions of the same figure, making a spike at the 39th harmonic. The low harmonics (including the "fundamental"--that is, the length of the piece) have almost nothing to do with it. On the other hand, Xenakis's very angular music shown in figure 9.28 has large-amplitude harmonics all across the field, high and low. The difference in the sound of these pieces is spectacular and the spectra show it.
In general, then, we are trying to do what electrical engineers do when they do a spectral analysis of sound. They do this very often when analyzing clarinet tones, for example, and to a lesser extent but much more intensively when processing CDs. Engineers know all about it, but music theorists have never applied it to melodic and harmonic analysis. Sooner or later, we will have to automate it so that it can read a score and make a mathematical analysis, but the geniuses who can do that aren't here yet.

If this is too technical or vague, let's talk again.
Charles

- I hope I have not put you off.

  Charles

- Mr. Madden,

  No, you have not put me off. I apologize for not responding to your last e-mail. The last few weeks have been incredibly busy for me as the deadline for my project is coming up soon along with a chamber ensemble peformance two days before. (And my flash drive with my 25-page rough draft on it mysteriously disappeard, so I have had to rewrite all of it.)

  Your explanation of the attractors did help, and although I don't have time to interlibrary loan the Lauwerier book, I was finally able to find an journal article online that is at a sufficiently elementary level for me to (mostly) understand it.

  The more I think about (or, as you put it, meditate on) the Fourier analysis, the more sense it makes to me. As I understand it now, the spectral analysis is showing the ?amplitudes?--I'm not sure if that is the correct term--of the various smaller "sound" curves that are combined to make the shape of the melodic line that I started with. From the examples in your book and the graphs that I have made, it seems to me that the lowest harmonics typically have higher amplitudes while the higher harmonics vary in amplitude. If (as in all the pieces I am analyzing) the higher harmonics have little to no activity (very little amplitude), this just tells me that the melodic line is "simple." Not necessarily simplistic, but less angular or random than some of the examples in your book, like the Xenakis. Or, as you put it, the Bach melodies I am studying have less turning and twisting, and therefore their sound curves are much simpler.

  Is this a correct summary statement?

  Thanks again for all your help.

  Jennifer Shafer

- At this point in the correspondence, Mr. Madden sent an e-mail (which I deleted) offering the loan of his copy of the Lauwerier book.

- -----Original Message-----
  From: Jennifer Shafer
  To: chasmadden@aol.com
  Sent: Fri, Oct 2, 2009 6:31 pm
  Subject: Re: Fourier Analysis
  Mr. Madden,
  That is a very generous offer, and I think I will take you up on it. Although I interlibrary loaned a copy of it last spring, I know I won't get it through the university library in time for it to be of any use to me now, so I really appreciate you offering to lend me yours. I will, of course, be more than happy to pay you for the shipping costs, and I will return the book to you as soon as I can. I appreciate your continuing interest in my project and your willingness to help me.

I guess you will need my shipping address:
Jennifer Shafer
ETBU Box 6-1338
1209 North Grove Street
Marshall, TX 75670
If you can't ship it directly to my ETBU box, just use the street address and the office here will deliver it to my post office box.
Once again, I greatly appreciate your help.
Jennifer Shafer

- OK

  I also wanted to say that I am using "meditation" in the sense of deep and prolonged thought.

  Also, that your computer technicians may have a system backup that contains your rough draft.

  Charles

- -----Original Message-----
  From: Jennifer Shafer
  To: chasmadden@aol.com
  Sent: Tue, Oct 6, 2009 6:36 pm
  Subject: Re: Fourier Analysis
  Mr. Madden,
  I received your book in the mail today. Thank you for loaning it to me.
  I checked with our IT department, and they were unable to recover my document, unfortunately. But I have it all rewritten now, and am moving forward with the rest of it.
  One more question: In the Hsu article where they are measuring the dimension of music ("Fractal Geometry of Music, published 1990), they have a dimension of 2.4184 for the Bach Invention No. 1. I have duplicated their procedure, and although I have a couple of slight deviations from their interval counts, overall I have the same results. However, what I cannot figure out is why they only use the intervals from 2 to 10 when they measure the dimension. I realize why they omit the 14 and 19 from the left hand--they are obviously not important, recurring intervals, and they omit 0 because they are using logarithms and 0 gives an error. But I can't figure out why they omit 1, 11, and 12--especially 1, since the minor seconds are a large percentage of the total intervals.
  I am attaching part of my file for invention no. 1. In the worksheet marked "Plot, Hsu", you can see the two charts: my version, including the intervals from 1 to 12, inclusive; and their version, including the intervals from 2 to 10, inclusive. When I do it their way, I get very close to their dimension, as you can see in the box to the left. My version seems like a more reasonable number for a dimension, but I would still like to figure out why they are omitting certain intervals. Any enlightment that you can give me would be greatly appreciated.
  Thanks.
  Jennifer Shafer

- Glad you got the book. I was getting worried, since the tracking information did not show it having arrived. You do not need to reimburse me. Just send it back when you're finished.

  Thanks for the worksheet. I couldn't understand it before, and disputed their value on page 157, but I will try to see what they are doing. Apparently you at least know how to set up the file and I'll study your setup.

  Our leaves were turning beautifully in the mountains, then it snowed and they all fell down. A very short season.

  Regards,

  Charles

- -----Original Message-----
  From: Jennifer Shafer
  To: chasmadden@aol.com
  Sent: Thu, Oct 8, 2009 9:08 am
  Subject: Re: Fourier Analysis

  Mr. Madden,
  Thanks for taking the time to look into it for me. Please let me know if you get it figured out. When I first started learning about fractals and their applications to music for a research paper my sophomore year, I came across this article, which I used as an example in my paper. I was able to duplicate their graph and their dimension using the information given in the article, and I never even thought about questioning why they didn't include all of the intervals. I guess I just figured that it was something I couldn't understand without having a deeper understanding of the topic. Now I am not sure how to apply their work to my own analyses . . . I have e-mailed my math advisor the same question, but I have not received a response yet.
  As I think I told you in my first e-mail, I am going to school in Texas. However, I don't think I ever told you that I am originally from southwest Wyoming. I miss the mountains and the ch anging seasons a lot when I am down here. It has been rainy and a little bit cooler here this last week, but my family has been getting snow while everything here is still green--the leaves don't usually change here until late October or early November. I definitely miss the cold weather that I am used to from home whiel I am in Texas for the fall and winter months!

Jennifer

- Would that be Evanston?

  Your Hsu graph looks different from the one in their article. It is cut off on the right and the bottom (yours is, too). That's where your missing 12, 14 and 19 are. Also, your version of their version has not picked up 1.35, which is interval 1. Maybe it's a problem of the range in the graph(?). I'd check that first.

  As you know, I did not agree with Hsu's value for the dimension. I never was able to understand their constant c. I'm not sure I would make an issue of it until I had a greater understanding of what they were doing. I have never seen a decent explanation of log-log graphs, although I'm sure there must be one somewhere. My basic question is a bit stupid: If the vertical scale can go negative (fractions), why not the horizontal too?

  It's a bit of a minefield arguing with the big boys. They know something I don't, and they don't want to help. I tried to open a dialog with Richard Voss about Gaussian white noise and he just yelled at me. Mandelbrot didn't even bother to answer when I asked about the validity of the Hurst exponent.

  Keep up the good work.

  Charles

- -----Original Message-----
  From: Jennifer Shafer
  To: chasmadden@aol.com
  Sent: Thu, Oct 8, 2009 2:39 pm
  Subject: Hsu graph
  No, actually not quite that far south. I am from a small town, Big Piney, which is slightly east and a good bit north of Evanston.
  I have often wondered why the graph is called a log-log plot when only one of the axes uses a logarithmic scale. I guess it is because the numbers that are being graphed on the other axis are logarithms anyway? But, if I understand it correctly, that is why the horizontal scale can't go negative--because you can't take a logarithm of any number less than or equal to 0. Is this correct?
  I didn't communicate well in my first e-mail or in the first file I sent you. I kept talking about them omitting the 1, 12, 14, and 19 intervals. I was referring to omitting them when they calculated the dimension, not from the graph, like I have (I hope) shown in the new file. I had deleted them from my graph so that I could get the trendline correct. I think this new file shows what I mean.
  Would you mind taking a look at the file I am attaching? It is almost an exact copy of the Hsu's graph. There are three differences that I am aware of. First, my x-axis goes out to 100. I cannot make it stop at 20; I guess because the x-axis is a logarithmic scale? I am not too concerned about that at this point. Secondly, I have graphed two series, some points of which are duplicated. Best I can tell, the series titled "All Intervals" duplicates their graph very closely (except for a few minor discrepancies (my third difference) in the two interval counts, which affect the graph only slightly). The second series, labelled "Intervals 2-10" is graphed with only the numbers for intervals 2-10, and I have used this series to draw the trendline. When I use this, my graph and my trendline looks like theirs except for the duplicated entries and a slight deviation at intervals 9 and 10, due to t he differences in the interval counts. The computed dimension, whi le not perfectly exact, is very close.
  I don't want to make an issue of it, since I guess I don't fully understand what they are doing (I don't know where their constant comes from, either). But since I can't find any reasoning for omitting those interval counts when computing the dimension, I am wondering whether I should include this method in my own analyses of the inventions and sinfonia (I could just leave it out), and if so, whether I should include all the intervals, or try to figure out their method of omitting some of the intervals. In the end I guess I have to decide this for myself, since it's my project, but what would you suggest?
  I have to say that I usually send these e-mails with some amount of apprehension--having questions for the big boys intimidates me, which is why I really appreciate your willingness to keep answering my stupid questions, or at least getting me on the right track to figuring them out.
  Jennifer

- Since I don't know how Hsu got that number, I hesitate to say anything about it. I don't know why any of the data points should be left out. Also, Hsu's number differs greatly from either yours or mine. It would be unwise to say anything other than "it needs further study." It's okay to admit that you cannot verify someone else's number. In fact, it may actually help another someone to tactfully say that.

  At this point, I am not entirely sure of the validity of using dimension in music. I think it's a good idea, but I need to put it on a firmer basis involving statistical analysis. Mandelbrot's Hurst exponent is a measure of dimension that modifies Brownian motion, which in turn is a measure of correlation, which is a statistic, and so on. It will take me several years to sort this out.

  It seems to me that the Fourier analyses are the most useful because they go the deepest and say the most to a practiced eye, but also require the most explanation. The orbit diagrams that relate to attractors are certainly more eyecatching and self-evident and should be included if only for effect.

  Hope that helps.

  Charles